Conformally Flat Condition (CFC)

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PennState
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IGC

Self-Gravity

Galaxy Star/Planet **AIC** **CCSNe** Neutron stars Black holes

Weak Field ⟵ ⟶ Strong Field

Conformally flat condition (CFC)
(Cordero-Carrion 2007)

Newtonian Gravity

- $\nabla^2 \phi = 4\pi\rho$
- Newtonian MHD
- Elliptic PDE
- Time step $\Delta t \sim \Delta x / v_{\mathrm{fluid}}$

(c.f. talk from David)

- Strong gravity
- Gravitational radiation subdominant (waveless approximation)
- Elliptic PDE
- GRMHD
- Time step $\Delta t \sim \Delta x / v_{\mathrm{fluid}}$

  Larger time step for $v_{\mathrm{fluid}} \ll c$

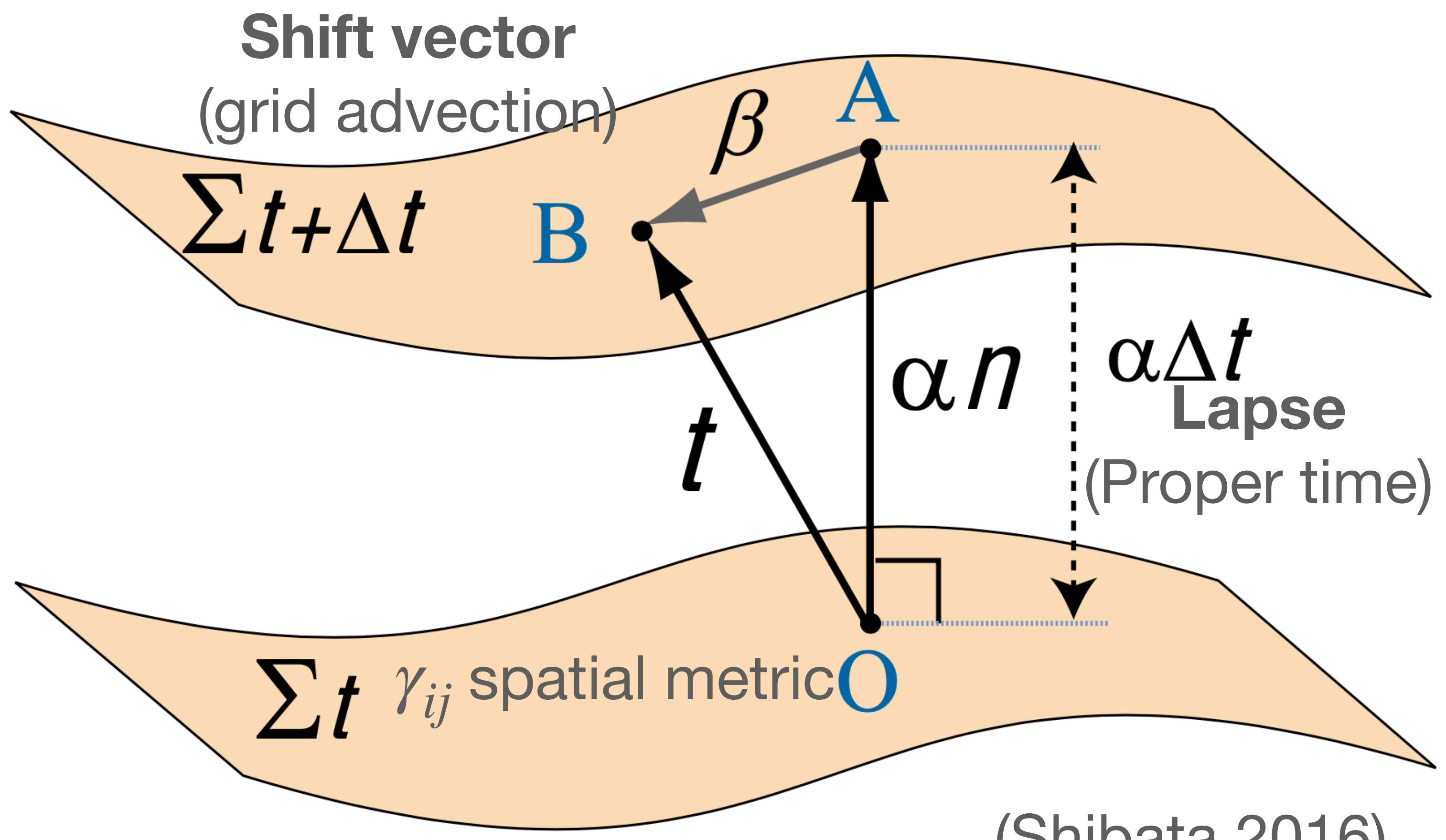
Numerical Relativity

- Einstein Equations $G_{ab} = 8\pi T_{ab}$
- Fully resolve gravitational radiation
- GRMHD
- Hyperbolic-(elliptic) PDEs
- Time step $\Delta t \sim \Delta x / c$

(c.f. talk from Manuela, Hengrui)

3+1 Decomposition

- normal vector $n^i = (1, -\beta^i)/\alpha \perp$ hypersurface Σ_t
  Spatial metric $\gamma_{ab} := g_{ab} + n_a n_b$
  $ds^2 = -\alpha^2 dt^2 + \gamma_{ij}\left(dx^i + \beta^i dt\right)\left(dx^j + \beta^j dt\right)$



(Shibata 2016)

- α : **Lapse**
  Choice for your **proper time** $\Delta\tau = \alpha\Delta t$ between next slice $\Sigma_{t+\Delta t}$
  β^i : **Shift vector**
  Choice for your spatial coordinate $x^i \to x^i - \beta^i \Delta t$ in next slice $\Sigma_{t+\Delta t}$

Gauge condition

- Extrinsic Curvature $K_{ij} \sim \partial_t \gamma_{ij} \Rightarrow$ conjugate pair (γ_{ij}, K_{ij}) similar to (x, p) in Classical Mechanics

- **Temporal** projection n^a
  Spatial projection $\gamma^a{}_b$

Decompose Stress-Energy Tensor T_{ab}

$E := n^a n^b T_{ab}$ Energy

$S_i := -n^a \gamma^b{}_i T_{ab}$ Momentum

$S_{ij} := \gamma^a{}_i \gamma^b{}_j T_{ab}$ Shear stress Tensor

3+1 Decomposition on Einstein Eq.

$G_{ab} = 8\pi T_{ab}$

to obtain evolution equations for (γ_{ij}, K_{ij})

ADM equations

D_i :covariance derivative for γ_{ij}

- Arnowitt-Deser-Misner (ADM) equations

- **Evolution Equations** for (γ_{ij}, K_{ij}) (12 equations)

$G_{ij} = 8\pi T_{ij}$

∂_t^2: Wave Eq.

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i$$

$$\partial_t K_{ij} = -D_iD_j\alpha + \alpha\left(R_{ij} - 2K_{ik}K^k{}_j + KK_{ij}\right)$$

$$-8\pi\alpha\left(S_{ij} - \frac{1}{2}\gamma_{ij}\left(S^k{}_k - \rho\right)\right) + \beta^k D_k K_{ij} + K_{ik}D_j\beta^k + K_{jk}D_i\beta^k$$

Weak fields Limit ($\alpha = 1, \beta^i = 0$ + Leading order term)

$$\partial_t \gamma_{ij} = -2K_{ij}$$

$$\partial_t K_{ij} = \frac{1}{2}\partial_k\partial^k\gamma_{ij} - 8\pi\left(S_{ij} - \frac{1}{2}\gamma_{ij}\left(S^k{}_k - E\right)\right)$$

$$\Rightarrow \Box\gamma_{ij} = 16\pi\left(S_{ij} - \frac{1}{2}\gamma_{ij}\left(S^k_k - E\right)\right)$$

Einstein Eq: $8\pi\left(T_{ij} - \frac{1}{2}g_{ij}T\right) = R_{ij} \sim \frac{1}{2}\Box\gamma_{ij}$

- **Constraint Equations** (4 equations)

$G_{00} = 8\pi T_{00}$ $\quad R + K^2 - K_{ij}K^{ij} = 16\pi E$ $\quad$ Hamiltonian (γ_{ij})

$G_{0i} = 8\pi T_{0i}$ $\quad D_j\left(K^{ij} - \gamma^{ij}K\right) = 8\pi S^i$ $\quad$ Momentum (K_{ij})

∂_t: Constraints between $(\gamma_{ij}, \partial_t\gamma_{ij})$

Maxwell Equations:

$$\partial_t E = \nabla \times B - 4\pi J$$

$$\partial_t B = -\nabla \times E$$

$$\nabla \cdot E = 4\pi\rho$$

$$\nabla \cdot B = 0$$

Recast Einstein equations into **Hyperbolic-Elliptic PDEs**

Evolution equations **conserve** the constraint equations

12 Evolution equations

-4 Gauge conditions

-4 Constraint equations

CFC

~~$= 4$ dynamical degrees of freedom~~

$(h_+, h_\times, \partial_t h_+, \partial_t h_\times)$

Conformal Transformation

$\tilde{D}_i$:covariance derivative for f_{ij}

- Factor out the main "**scale**" of spatial metric $\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij}$
  ψ: conformal factor
  $\tilde{\gamma}_{ij}$: conformal spatial metric

Schwarzschild BH in isotropic coordinate:

$$ds^2 = -\left(\frac{1 - M/(2r)}{1 + M/(2r)}\right)^2 dt^2 + \left(1 + \frac{M}{2r}\right)^4 (dr^2 + r^2 d\Omega)$$

- **Conformally Flat**: $\tilde{\gamma}_{ij} = f_{ij}$, $\gamma_{ij} = \psi^4 f_{ij}$ (isotropic coordinate)

- **Exact** in spherically symmetric spacetime
  $\Rightarrow$ Ideal for systems close to spherical symmetry (CCSNe, NS)

Exercise: Transform TOV solutions (radial gauge) to isotropic coordinate

- Conformal Transformation on extrinsic curvature $\tilde{A}_{ij} = \psi^{-4}\left(K_{ij} - \frac{1}{3}\gamma_{ij}K\right)$ (traceless part of K_{ij})

- Hamiltonian constraint: $R + K^2 - K_{ij}K^{ij} = 16\pi E$
  $R \sim R_\psi + \tilde{R} \sim \tilde{\Delta}\psi$

Hamiltonian constraint $\tilde{\Delta}\psi = -2\pi\psi^5 E - \frac{1}{8}\psi^5 \tilde{A}_{ij}\tilde{A}_{ij}$ (set $K = 0$ for the reason shown in the next slide)

Maximal-slicing condition

Recall $\partial_t K_{ij} = -D_i D_j \alpha + \ldots$

- Consider the trace of $\partial_t K_{ij}$ equation: $\partial_t K = -\Delta\alpha + \alpha\left(K_{ij}K^{ij} + 4\pi\left(E + S^k{}_k\right)\right) + \beta^i D_i K$

- This gives an elliptic equation for α: $\Delta\alpha = -\partial_t K + \alpha\left(K_{ij}K^{ij} + 4\pi\left(E + S^k{}_k\right)\right) + \beta^i D_i K$

- To avoid hyperbolic evolution on the metric,
  we need to impose **gauge conditions** to enforce equilibrium $\partial_t K = 0$

- **Maximal-slicing condition**: $K = 0 = \partial_t K$ $\quad\Rightarrow \Delta\alpha = \alpha\left(A_{ij}A^{ij} + 4\pi\left(E + S^k{}_k\right)\right)$

- **Singularity avoidance property**:
  freely-falling observer (coordinate) takes **infinite** coordinate time t to reach BH singularity
  $\Rightarrow$ allow stable spacetime evolution with BHs

Maximal slicing condition: $\tilde{\Delta}\left(\alpha\psi\right) = \alpha\psi\left(\frac{7}{8}\tilde{A}_{ij}\tilde{A}^{ij} + 2\pi\left(E + 2S^k{}_k\right)\right)$

Conformal Thin Sandwich (CTS)

- We imposed $\tilde{\gamma}_{ij} = f_{ij}$ at $t = 0$. How do we make sure $\tilde{\gamma}_{ij}$ remains flat in time evolution?
- Set $\partial_t \tilde{\gamma}_{ij} = 0$ is a **gauge choice,** not consequence of the Einstein equations

Commonly used in constructing **initial data** for binary neutron stars/BH to enforce quasi-equilibrium $\partial_t \tilde{\gamma}_{ij} = 0$

- Recall the evolution equation: $\partial_t \tilde{\gamma}^{ij} = 2\alpha \tilde{A}^{ij} - \left(\tilde{D}^i \beta^j + \tilde{D}^j \beta^i - \frac{2}{3} \tilde{\gamma}^{ij} \tilde{D}_k \beta^k \right) = 0$
- $\tilde{A}^{ij} = \frac{1}{2\alpha} \left(\tilde{D}^i \beta^j + \tilde{D}^j \beta^i - \frac{2}{3} \tilde{\gamma}^{ij} \tilde{D}_k \beta^k \right) = \frac{1}{2\alpha} \left(\mathscr{L} \beta \right)^{ij}$

Conformal Thin Sandwich (CTS) decomposition

- $\tilde{A}^{ij}$ (6 DoF) $\to \beta^i$ (3 DoF), we can fix the remaining 3 DoF by **momentum constraint**: $D_j \left(K^{ij} - \gamma^{ij} K \right) = 8\pi S^i \; \sim \tilde{\Delta} \beta^i$

CTS decomposition $\tilde{\Delta} \beta^i + \frac{1}{3} \tilde{D}^i \tilde{D}_j \beta^j - 2\alpha \tilde{A}^{ij} \tilde{D}_j \ln \left(\alpha \psi^{-6} \right) = 16\pi \alpha f^{ij} S_j$

The original CFC Scheme

- Isenberg-Wilson-Mathews formulation (the original CFC)
- $\tilde{\gamma}_{ij} - f_{ij} = 0 = \partial_t \tilde{\gamma}_{ij}$, solve for $(\psi, \alpha\psi, \beta^i)$

- $\tilde{\Delta}\psi = -2\pi E\psi^5 - \frac{1}{8}\tilde{A}_{ij}\tilde{A}^{ij}\psi^5$ **Hamiltonian Constraint**
- $\tilde{\Delta}(\alpha\psi) = \alpha\psi\psi^4\left[2\pi\left(E + 2S^k{}_k\right) - \frac{7}{8}\tilde{A}_{ij}\tilde{A}^{ij}\right]$ **Maximal Slicing**
- $\tilde{\Delta}\beta^i + \frac{1}{3}\tilde{D}^i\tilde{D}_k\beta^i + \tilde{A}^{ij}\tilde{D}_j \ln\left(\alpha\psi^{-5}\right) = 16\pi f^{ij}\alpha S_j$ **Momentum Constraint**
- $\tilde{A}^{ij} = \frac{1}{2\alpha}\left(\mathscr{L}\beta\right)^{ij} = \frac{1}{2\alpha}\left(\tilde{D}^i\beta^j + \tilde{D}^j\beta^i - \frac{2}{3}f^{ij}\tilde{D}_k\tilde{D}^k\beta^k\right)$ **CTS** $\partial_t\tilde{\gamma}_{ij} = 0$

- All the equations are coupled together, so we need to iterate until (ψ, α, β^i) converge (**expensive**!).
- Non-linear equation for ψ that suffered from a **non-uniqueness** problem!

Non-uniqueness Problem

(Pfeiffer & York 2005)

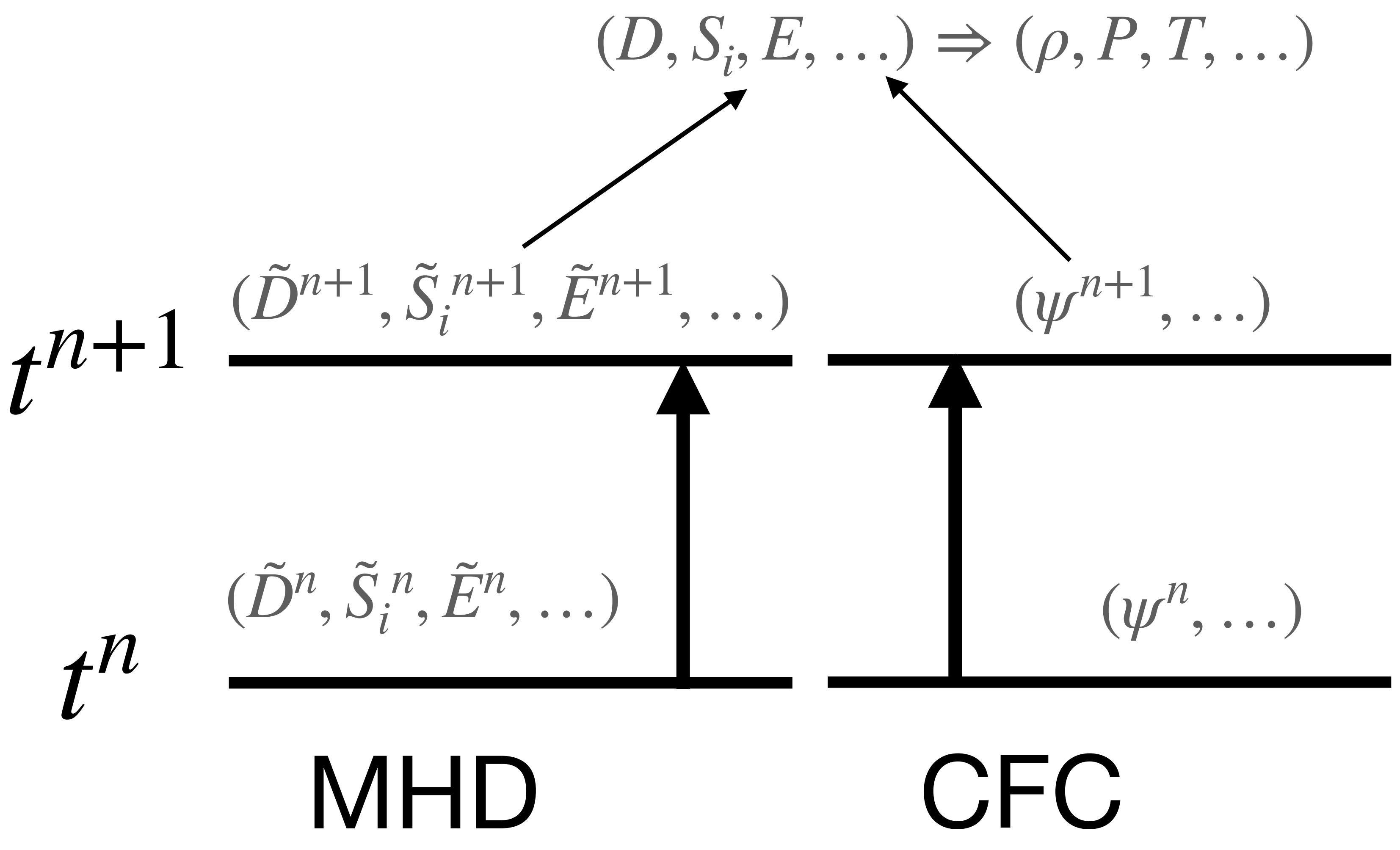
- First found in CTS scheme
- For scalar elliptic equations in form $\Delta u = -hu^p$
- Local uniqueness only for $h \cdot p < 0$

$$\tilde{\Delta}\psi = -2\pi E\psi^5 - \frac{1}{8}\tilde{A}_{ij}\tilde{A}^{ij}\psi^5$$

$$\tilde{\Delta}\psi = -2\pi\tilde{E}\psi^{-1} \boxed{-\frac{1}{8}\tilde{A}_{ij}\tilde{A}^{ij}\psi^5}$$

$\frac{5}{8}\tilde{A}_{ij}\tilde{A}^{ij} > 0$

Conserved variables $\tilde{D} := \sqrt{\gamma}D = \psi^6 D$



The CFC scheme suffered from the **non-uniqueness** problem
We need a reformulation

CTT decomposition

- Consider another scaling for extrinsic curvature $\hat{A}^{ij} := \psi^{10} A^{ij}$ $\Rightarrow \tilde{A}_{ij}\tilde{A}^{ij}\psi^{5} = \hat{A}_{ij}\hat{A}^{ij}\psi^{-7}$
- Reformulate Momentum Constraint $\tilde{D}_j \hat{A}^{ij} = 8\pi f^{ij}\psi^6 S_j = 8\pi f^{ij}\tilde{S}_j$

Solve the non-uniqueness problem!

- Conformal Transverse-Traceless (CTT) decomposition
$\hat{A}^{ij} = \left(\mathscr{L}X\right)^{ij} + \hat{A}^{ij}_{TT}$, where $\tilde{D}_j \hat{A}^{ij}_{TT} = 0$

Longitudinal Transverse -Traceless

For vector $\mathbf{F} = \nabla\phi + \nabla\times\mathbf{A} \Rightarrow \nabla\cdot\mathbf{F} = \Delta\phi$

- We can further assume $\hat{A}^{ij}_{TT} = 0$ because
  1. For quasi-spherical system, $\hat{A}^{ij}_{TT} \ll \tilde{\gamma}^{ij} - f^{ij} \sim \mathcal{O}(e^2/c^4)$ (Cordero-Carrion 2007)
  2. In weak-field regime, $\hat{A}^{ij}_{TT} \sim \partial_t h_+, \partial_t h_\times$ carries information about gravitational radiation

CTT decompostion: $\tilde{\Delta}X^i + \frac{1}{3}\tilde{D}^i\tilde{D}_j X^j = 8\pi f^{ij}\tilde{S}_j$, $\hat{A}^{ij} = \left(\mathscr{L}X\right)^{ij}$

$\tilde{\Gamma}^i$-freezing gauge condition

- In the original CFC, we use CTS decomposition to fix $\tilde{A}^{ij}$ $(\partial_t \tilde{\gamma}^{ij} = 0)$
  Now, we use momentum constraints to fix $\hat{A}^{ij} = \psi^{-6}\tilde{A}^{ij}$ (CTT decomposition)

- Can we still impose the equilibrium condition $(\partial_t \tilde{\gamma}^{ij} = 0)$?
  Sort of… we still have 3 DoF left (**gauge condition** for β^i!)

- Recall $\partial_t \tilde{\gamma}^{ij} = 0 = 2\alpha\psi^{-6}\hat{A}^{ij} - \left(\mathscr{L}\beta\right)^{ij} \Rightarrow$ CTT decomposition: $\tilde{\gamma}^{ij} = \tilde{\gamma}^{ij}_{L} + \tilde{\gamma}^{ij}_{TT}$ (Longitudinal; Transverse-Traceless)

- Impose **gauge condition** to enforce the **longitudinal** part = 0 $(\tilde{D}_j \tilde{\gamma}^{ij} = 0 = \tilde{\Gamma}^i)$

- Assumption made here: $\partial_t \tilde{\gamma}^{ij}_{TT} = 0 \Leftarrow$ associate with the gravitational wave DoF. Waveless!

$\tilde{\Gamma}^i$-freezing/Minimal Distortion Gauge: $\tilde{\nabla}^2\beta^i + \frac{1}{3}\tilde{\nabla}^i\left(\tilde{\nabla}_j\beta^j\right) = 16\pi\alpha\psi^{-6}f^{ij}\tilde{S}_j + 2\hat{A}^{ij}\tilde{\nabla}_j\left(\alpha\psi^{-6}\right)$

Extended CFC (xCFC)

(Cordero-Carrion 2007)

- Conformally Flat $\tilde{\gamma}^{ij} = f^{ij}$
- Approximation: $\hat{A}^{ij}_{TT} = 0 = \partial_t \tilde{\gamma}^{ij}_{TT}$ (waveless)
- $\tilde{\nabla}^2 X^i + \frac{1}{3}\tilde{\nabla}^i\left(\tilde{\nabla}_j X^i\right) = 8\pi f^{ij}\tilde{S}_j$ Momentum Constraint
- $\tilde{\nabla}^2 \psi = -2\pi \tilde{E}\psi^{-1} - \frac{1}{8}\hat{A}_{ij}\hat{A}^{ij}\psi^{-7}$ Hamiltonian Constraint
- $\tilde{\nabla}^2(\alpha\psi) = (\alpha\psi)\left[2\pi\left(\tilde{E} + 2\tilde{S}^k{}_k\right)\psi^{-2} + \frac{7}{8}\hat{A}_{ij}\hat{A}^{ij}\psi^{-8}\right]$ Maximal Slicing
- $\tilde{\nabla}^2 \beta^i + \frac{1}{3}\tilde{\nabla}^i\left(\tilde{\nabla}_j \beta^j\right) = 16\pi\alpha\psi^{-6} f^{ij}\tilde{S}_j + 2\hat{A}^{ij}\tilde{\nabla}_j\left(\alpha\psi^{-6}\right)$ Minimal Distortion

- **Non-uniqueness** problem solved
- No downstream dependence. We can just solve the metric in **consecutive** order.
- Always satisfy **constraints** and some **equilibrium** conditions for metric. $\Rightarrow$ Every step is an **initial data** for z4c

Solving Vector Equations

- Solve Vector equations for X^i and β^i : $\nabla^2 V^i + \frac{1}{3}\nabla^i\left(\nabla_j V^j\right) = \mathcal{S}^i$
- Three equations for V^x, V^y, V^z coupled via $\nabla^i\left(\nabla_j V^j\right)$
- Contain mixed derivatives (e.g. $\partial_x \partial_y X^i$)
  $\Rightarrow$ require edge ghost cells communication.
- Decompose V^i into a vector P^i + a scalar η (Oohara 1997, Shibata 1999)
  $$V^j = f^{ij}\left[\frac{7}{8}P_i - \frac{1}{8}\left(\partial_i \eta + x^k \partial_i P_k\right)\right]$$

Decompose vector equations: $\nabla^2 V^i + \frac{1}{3}\nabla^i\left(\nabla_j V^j\right) = \mathcal{S}^i \Rightarrow$ $\begin{aligned} \nabla^2 P^i &= \mathcal{S}^i \\ \nabla^2 \eta &= -x_i \mathcal{S}^i \end{aligned}$

Task flows for CFC

Update GRMHD conservative variables $(\tilde{D}, \tilde{S}_i, \tilde{\tau}, \ldots)$

Solve for X: $\nabla^2 X^i + \frac{1}{3}\nabla^i\left(\nabla_j X^j\right) = 8\pi f^{ij}\tilde{S}_j$

Reconstruct $\hat{A}^{ij} = \nabla^j X^i + \nabla^i X^j - \frac{2}{3} f^{ij}\nabla_k X^k$

Solve for ψ: $\nabla^2\psi = -2\pi\tilde{E}\psi^{-1} - \frac{1}{8}\hat{A}_{ij}\hat{A}^{ij}\psi^{-7}$

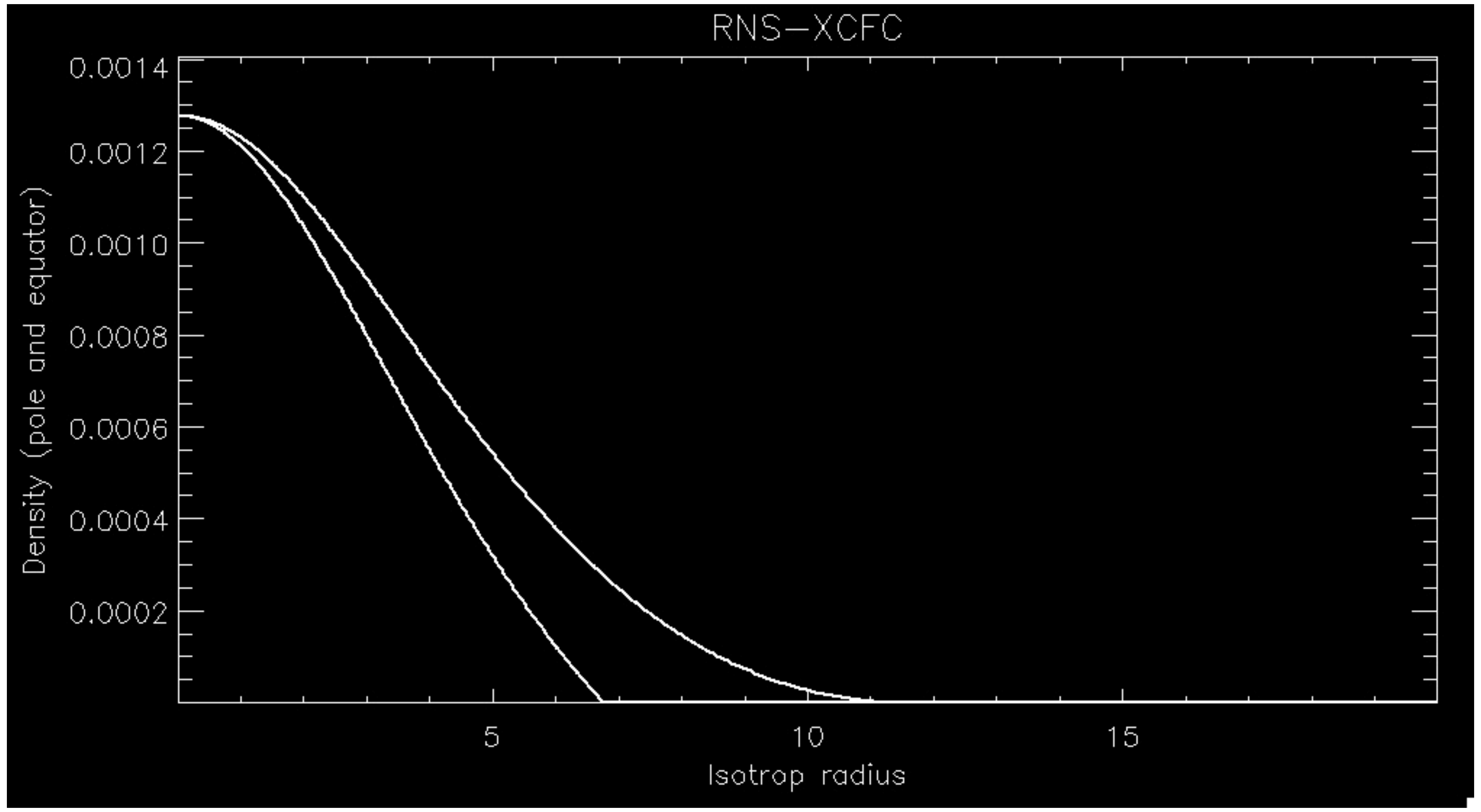
Obtain $(D, S_i, \tau, \ldots) \Rightarrow$ Primitive recovery $(\rho, P, T, \ldots) \Rightarrow$ Evaluate $\tilde{S}$

Solve for α: $\nabla^2(\alpha\psi) = (\alpha\psi)\left[2\pi\left(\tilde{E} + 2\tilde{S}\right)\psi^{-2} + \frac{7}{8}\hat{A}_{ij}\hat{A}^{ij}\psi^{-8}\right]$

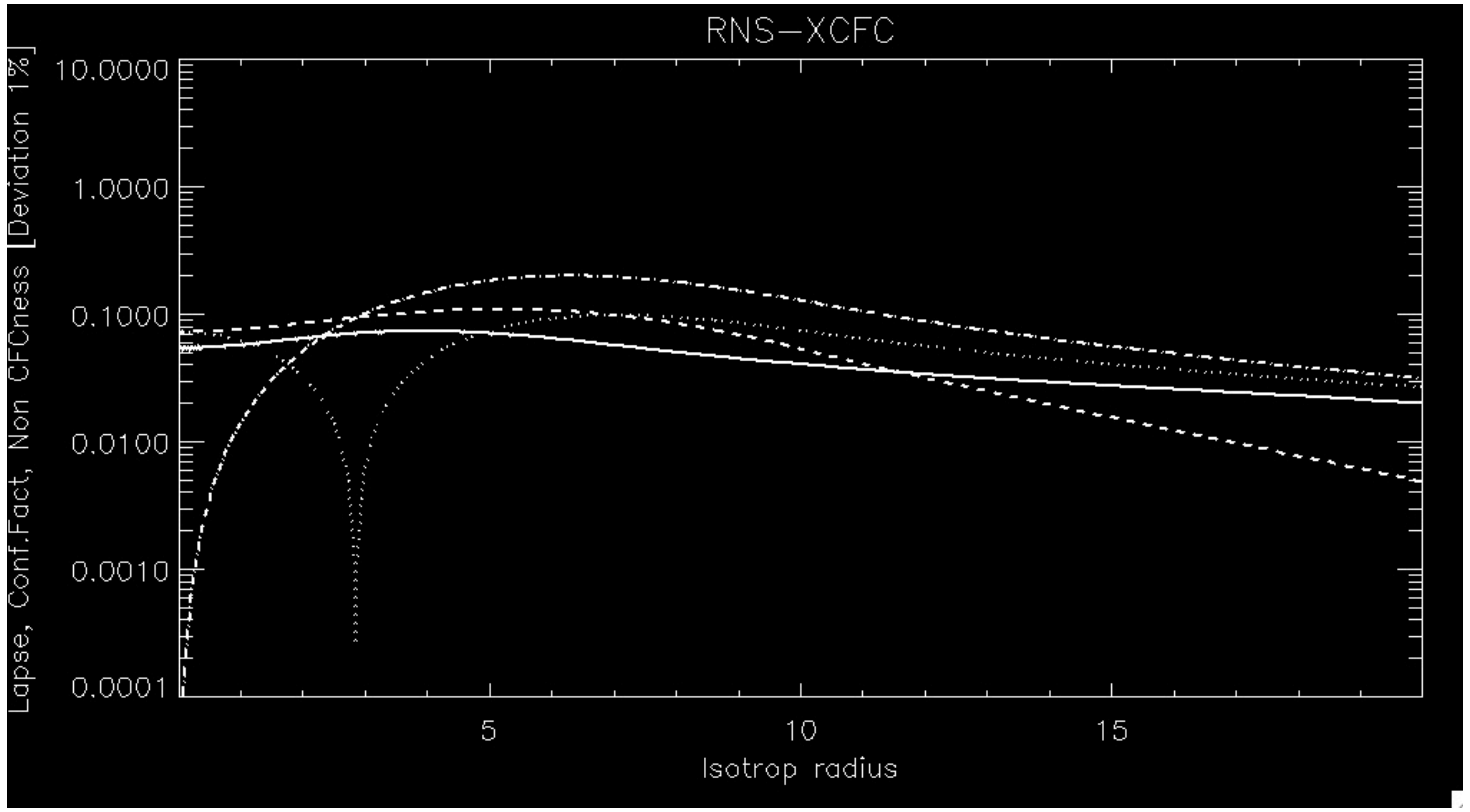
Solve for β^i: $\nabla^2\beta^i + \frac{1}{3}\nabla^i\left(\nabla_j\beta^j\right) = 16\pi\alpha\psi^{-6} f^{ij}\tilde{S}_j + 2\tilde{A}^{ij}\nabla_j\left(\alpha\psi^{-6}\right)$

Application: Neutron Star

- Comparison between **initial data** from **XNS** (xCFC) and **RNS** (full GR) for rotating NS



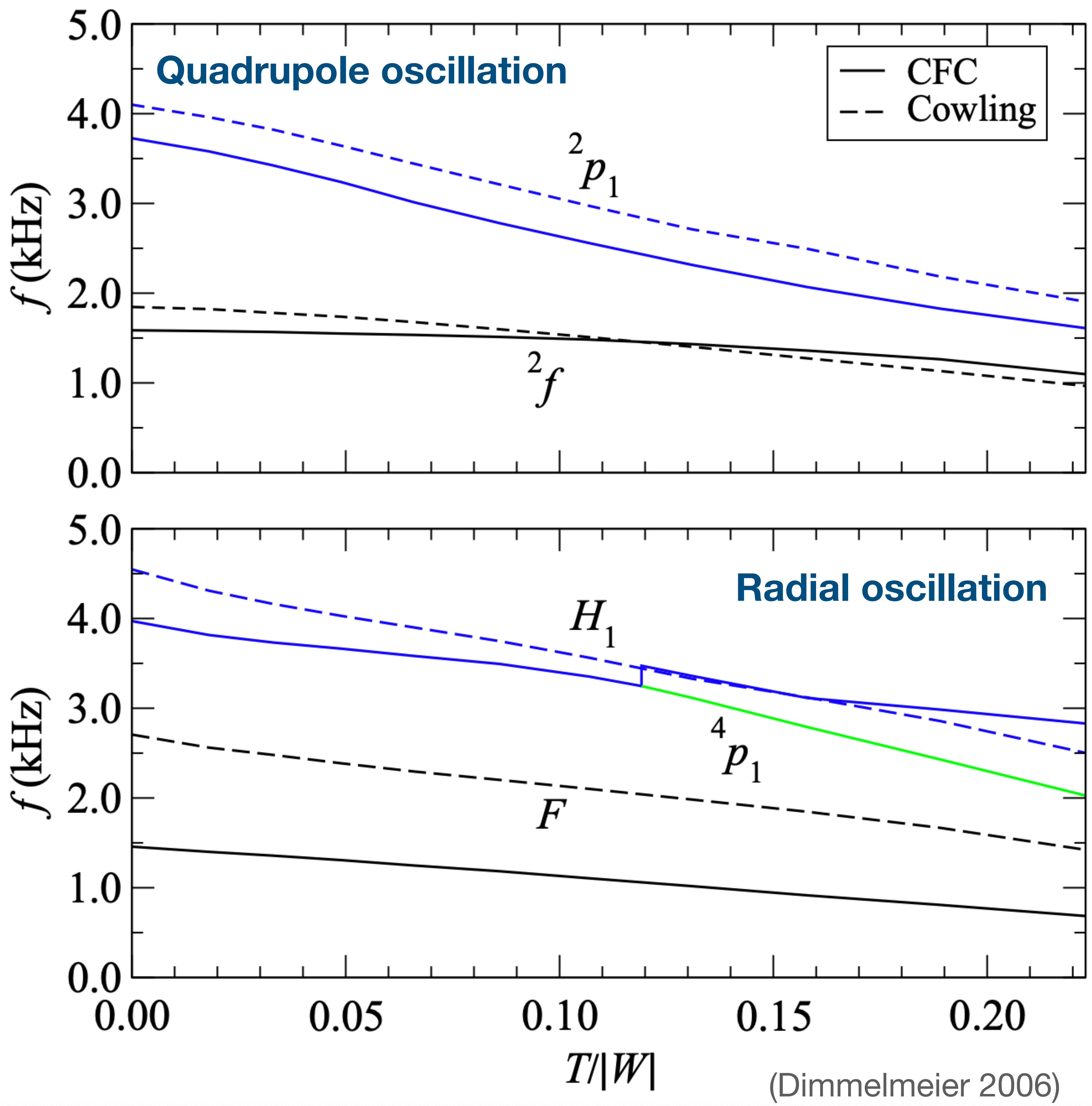
Polar and equatorial density profile
XNS (dash) vs RNS (solid)



Metric quantities deviate by $\lesssim 0.1\,\%$

Application: Neutron Star oscillation

- Neutron star oscillation mode in xCFC vs Cowling (stationary spacetime)
- Deviation up to a factor of 2
- Solving self-gravity is **necessary**
  ⇒ Important for CCSNe simulations to estimate GW frequency from PNS.



(Dimmelmeier 2006)

xCFC in AthenaK (Upcoming)

- Summer school is about initiating projects on AthenaK
  Start the development a few days ago (**Claude** assisted) c.f (Hengrui talk on AI)

- Thanks to David's multigrid solver and gravity module.

- Evolving a TOV star (mesh block 32^3)

- 100% Not **bug-free**

- **TODO:** debug & testing