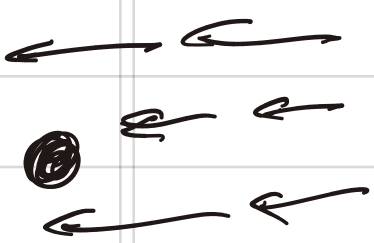
Dust and Gas interactions in Disks

Key quantity: the stopping time - how much time does it take for particle to come to rest due to gas drag.



particle frame — gas particles hit particle and slow it down — this is a non conservative force, like friction (unlike gravity)

$$t_{stop} \equiv \frac{m \Delta v}{F_{drag}}$$

m is mass of particle, $\Delta v = v_g - v$ relative velocity, F_{drag} is force due to drag.

Physical explanation: numerator = momentum exchanged
denominator = force to create ↑

So, t_{stop} is timescale over which $m\Delta v \to 0$ due to F_{drag}.

If particle size << ℓ_{mfp} of gas particles and particle is spherical, we can rewrite this as

$$t_{stop} = \frac{a \rho_s}{\rho_g v_{th}},$$

a - particle size, ρ_s - solid density of particle

ρ_g - gas density

$v_{th} \sim c_s$ ← sound speed.

in uniform gas, this is constant.

Simple example:

$$m\ddot{x} = m\dot{v} = F_{drag} = \frac{m\Delta v}{t_{stop}} \Rightarrow \dot{v} = \frac{\Delta v}{t_{stop}}$$

If gas is stationary, $\Delta v = -v$

$$\hookrightarrow \frac{dv}{dt} = -\frac{v}{t_{stop}} \Rightarrow \int_{v_0}^{v(t)} \frac{dv}{v} = -\int_0^t \frac{1}{t_{stop}} dt$$

Solution:

exp. decay due to drag →

$$\boxed{v(t) = v_0 e^{-t/t_{stop}}}$$

exp decay

Define new quantity for particles in rotating disk

↳ $St = t_{stop}\Omega$, Ω - Keplerian frequency
$\left(\Omega = \sqrt{GM/r^3}\right)$

Describes how well coupled to the gas the dust is

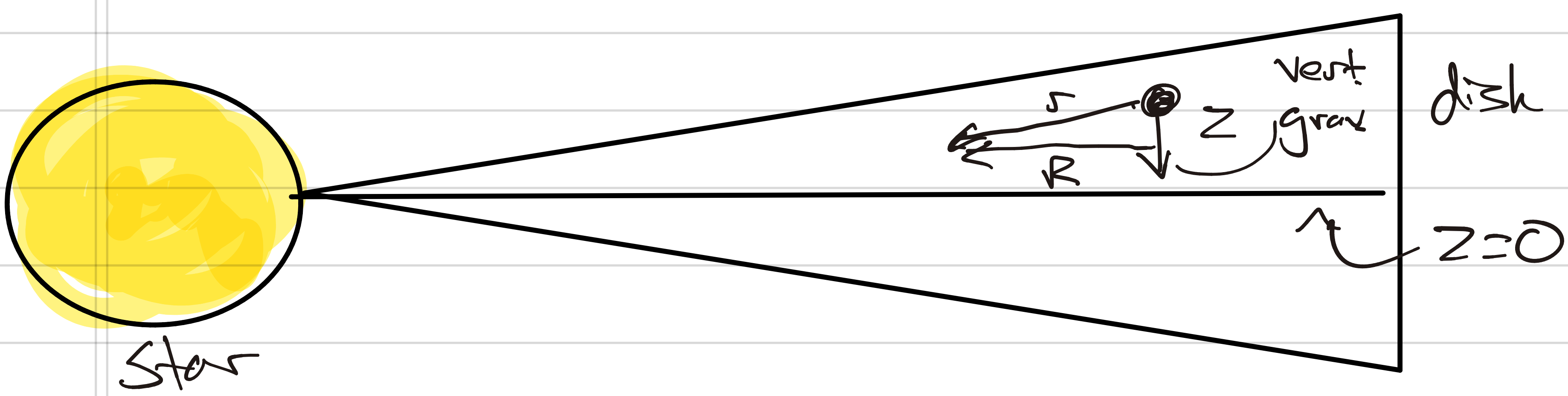
$St \ll 1 \Rightarrow t_{stop} \ll \Omega^{-1} \Rightarrow$ Strong coupling $\rightarrow$ couples faster than orbital time

$St \gg 1 \Rightarrow t_{stop} \gg \Omega^{-1} \Rightarrow$ weak "

↳ many orbits to "slow down"

$St \sim 1 \Rightarrow$ marginal coupling } will be the most interesting region dynamically

Now, consider particle in disk at some height z



Equation of motion in z: $\ddot{z} = \underbrace{-z\Omega^2}_{\text{vert. grav.}} - \frac{\dot{z}}{t_{stop}}$

If no drag ($t_{stop} \to \infty$)

↳ $\ddot{z} = -z\Omega^2$ harmonic oscillator! particle oscillates about midplane at the same rate it orbits

inclined orbit.

Ω is the rate

$\ddot{z} = -z\Omega^2 - \dot{z}/t_{stop}$ ← damped harmonic oscillator!

For $St \ll 1$, very quickly damped

Can find terminal velocity $\boxed{v_{sett} = -z\Omega St \quad (St \ll 1)}$

- This implies all dust settles onto midplane eventually.
- However, even a little turbulence in the gas can stir things up considerably.
- If you solve an equation for particles that include a turbulent diffusion

$$D \sim \delta v_z^2 \Omega^{-1}$$

Represents vertical velocity kicks over a Ω^{-1} timescale $\Rightarrow$ prevents complete settling.

Can relate this to a dimensionless quantity

$$\alpha \sim \delta v^2 / c_s^2$$

This assumes the same turbulence causing accretion is stirring up particles $\rightarrow$ may not be true in general!

$$D \sim \alpha c_s H$$

When accounting for all this, one can calculate the scale height of particles assuming a Gaussian profile.

$$\boxed{H_p = H \sqrt{\frac{\alpha}{St + \alpha}}}$$

Let's consider two limits:

$$St \gg \alpha \Rightarrow H_p \simeq H \sqrt{\frac{\alpha}{St}} \ll H$$

So, particles are largely settled.

If $St \ll \alpha$, $H_p = H\sqrt{\frac{\alpha}{St+\alpha}} \simeq H\sqrt{\frac{\alpha}{\alpha}} = H$

So, for strong turbulence or small particles, the scale height of particles approaches H.

* The scale height of particles can tell you about α if St is known

---

Radial Drift

Orbital equilibrium (u – gas, v – particles):

$$\frac{v_\phi^2}{r} = \frac{GM}{r^2} \qquad \frac{u_\phi^2}{r} = \frac{GM}{r^2} + \frac{1}{\rho}\frac{dP}{dr}$$

Since $\frac{dP}{dr} < 0$, $u_\phi < v_\phi = v_K \Rightarrow$ gas orbits more slowly.

By how much?

uses $P = \rho c_s^2$ (ideal gas)

$$\frac{1}{\rho}\frac{dP}{dr} = \frac{c_s^2}{P}\frac{dP}{dr} = \frac{c_s^2}{r}\frac{d\ln P}{d\ln r}$$

Assume power law for pressure (approx. consistent w/ observations)

$$P \propto r^{-n} \Rightarrow \frac{d\ln P}{d\ln r} = -n$$

$$\hookrightarrow \frac{u_\phi^2}{\cancel{r}} = \frac{v_K^2}{\cancel{r}} - \frac{c_s^2}{\cancel{r}} n \Rightarrow \qquad u_\phi = v_K(1-\eta)^{1/2}$$

$$\eta \equiv n\, c_s^2/v_K^2$$

$$\frac{c_s^2}{v_k^2} = \frac{\cancel{\Omega^2} H^2}{\cancel{\Omega^2} r^2} = (0.05)^2$$

Also, from observations, $P \propto r^{-5/2} \Rightarrow n = 5/2$

↳ so $\eta = \frac{5}{2}(0.05)^2 = 0.006 \Rightarrow \boxed{u_\phi = 0.997 v_k}$

At 1 AU, $\underline{v_\phi - u_\phi = 100 \text{ m/s}}$

Not very much of a difference!

Can take equation of motion in radial direction to get v_r

For sake of time, the (approx) solution is

$$\boxed{\frac{v_r}{v_k} \approx \frac{-\eta}{St + St^{-1}}}$$

(Assumes particle mass negligible)

This is a crucial assumption which we will return to later

Take limits,

$$St \gg 1 \Rightarrow \frac{v_r}{v_k} \approx -\eta/St \ll 1$$

$$St \ll 1 \Rightarrow \frac{v_r}{v_k} \approx -\eta\, St \ll 1$$

$$St \sim 1 \Rightarrow \frac{v_r}{v_k} \approx -\eta$$

Max radial drift at $St \sim 1$!

What is drift timescale for $St \approx 1$?

$$T_{drift} \approx r/|v_r| = \frac{r}{\eta v_K} = \frac{1}{\eta \Omega}$$
$$= \frac{2\pi}{2\pi \eta \Omega} = \frac{1}{2\pi \eta} T_{orb.}$$

If $\eta \approx 0.006$,
$$T_{drift} \approx 30\, T_{orb}$$

At 1AU, only takes 30 years for particles to drift inward!

Okay, so what's happening physically?

Move to particle's frame

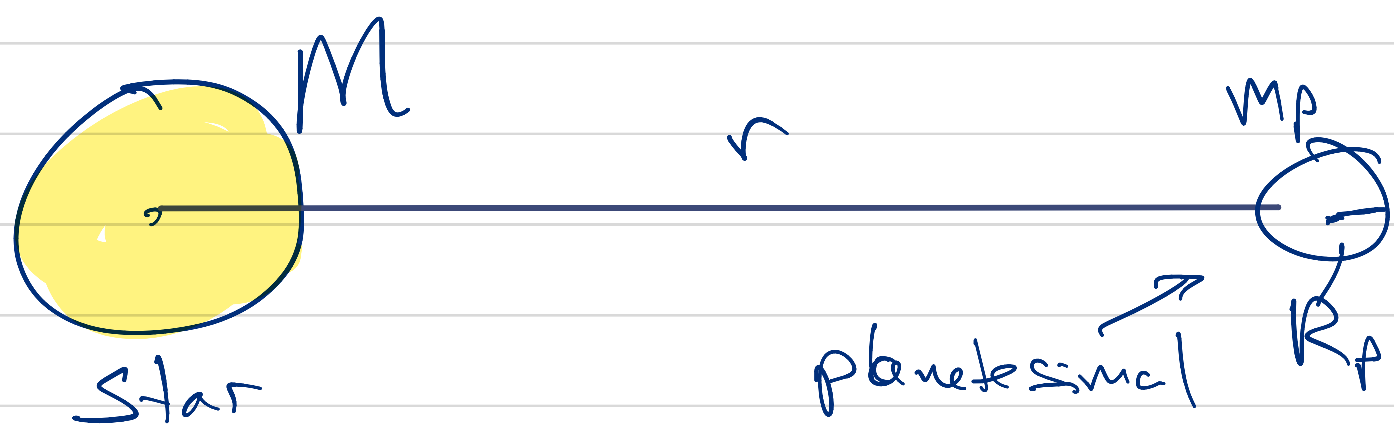
Two limits:

$St >> 1$: Gas orbits more slowly due to pressure gradient. Causes headwind on particles $\Rightarrow$ they lose ang. mom.

$St << 1$: Particles are "forced" to orbit w/ gas but gas is sub-Kep. So centrifugal accel of particles isn't strong enough and they drift inward

Planetesimal Formation

- The key to planetesimal formation lies w/ gravitational collapse.
- How does this work? Have to consider Roche density!

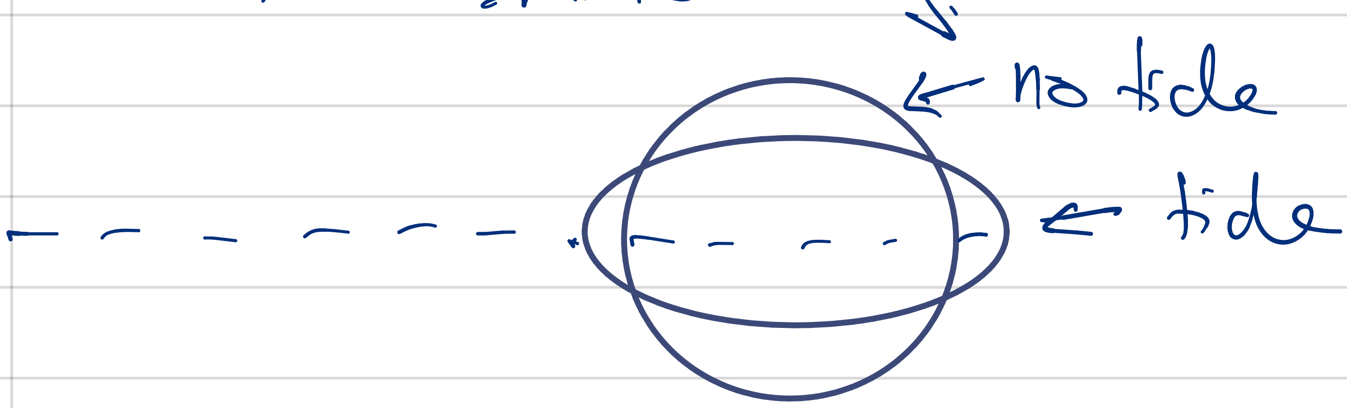


Tidal acceleration:

$$a_t = -\frac{GM}{(r+R_p)^2} + \frac{GM}{r^2}$$

If we assume $r \gg R_p$, can write this as $a_t \simeq \frac{2GMR_p}{r^3}$

Represents the force of "stretching" across the smaller body.



Need to balance this against self grav. of planetesimal.

$$a_s = \frac{GM_p}{R_p^2}$$

Condition for collapse:

$$a_s > a_t \implies \frac{GM_p}{R_p^2} > \frac{2GMR_p}{r^3}$$

$$\Leftrightarrow \quad r^3 > \left(\frac{M}{M_p}\right) R_p^3$$

Assuming uniform density $\bar{\rho}$, this becomes

$$\bar{\rho} > \frac{M}{4/3 \pi r^3}$$

More accurate calculation gives:

$$\boxed{\bar{\rho} > \rho_R \equiv 3.5 \frac{M}{r^3}}$$

This is the Roche density

Easier to collapse if:

(1) star has lower mass

or

(2) planetesimal is farther away

(or both)

- But Goldreich-Ward mechanism doesn't work.

- Why not?

$$Q_{peb} = \frac{\sigma_p \Omega}{\pi G \Sigma_{peb}} \qquad \Sigma_{peb} = 0.01 \Sigma_g$$

dust to gas ratio.

This means σ_p must be small to still allow $Q_{peb} < 1$

With typical disk conditions, $\sigma_p \approx 10 \frac{cm}{s}$ is necessary for collapse

$H_p = \frac{\sigma_p}{\Omega}$ (just like $H_g = c_s/\Omega$)

$\sigma = 10\ cm/s \Rightarrow H_p/H_g = 10^{-4}$

Use $H_p = H_g \sqrt{\frac{\alpha}{St + \alpha}}$ and assume $St = 10^{-3}$ (cm size pebbles at 1AU)

Solve for α:

$$\alpha \sim 10^{-11}!$$

Gas would need to be almost purely non-turbulent.