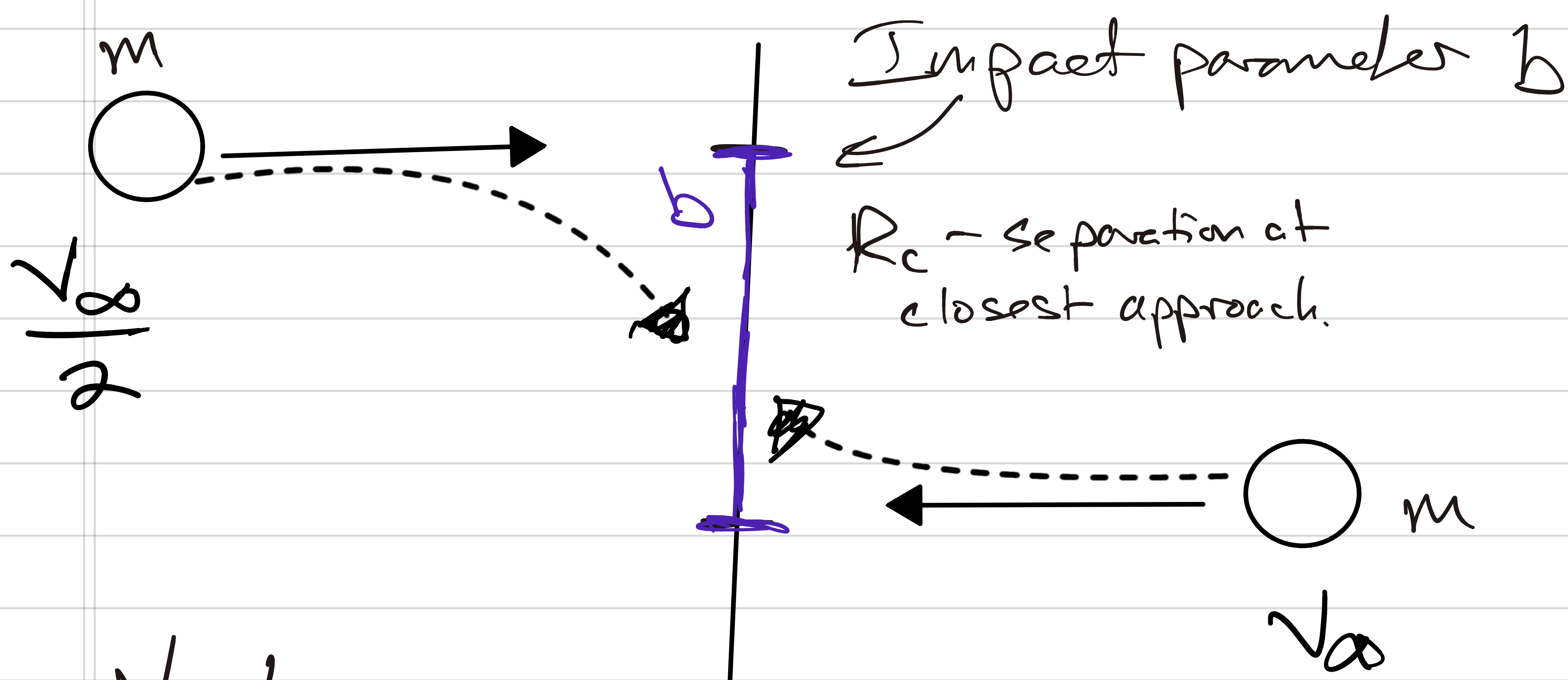
Planetary Embryos



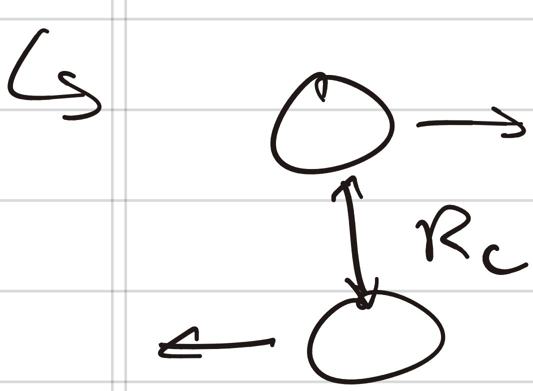
$V_\infty/2$ so that total rel. vel. at inf. is V_∞

Total energy cons.: $\frac{1}{2}m\left(\frac{V_\infty}{2}\right)^2 + \frac{1}{2}m\left(\frac{V_\infty}{2}\right)^2$

$$= mV_{max}^2 - \frac{Gm^2}{R_c}$$

$$\frac{1}{4}mV_\infty^2 = mV_{max}^2 - \frac{Gm^2}{R_c}$$

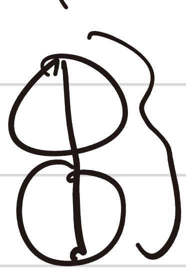
No radial motion at closest approach



$\Rightarrow$ $V_{max} = \frac{1}{2}\frac{b}{R_c}V_\infty$ ← from cons. of ang. mom.

Combining these two equations:

$$\frac{1}{4}mV_\infty^2 = m\frac{1}{4}\frac{b^2}{R_c^2}V_\infty^2 - \frac{Gm^2}{R_c}$$



R_s – combined radius of bodies

If $R_s > R_c$, largest value of b such that collision:

$$b^2 = R_s^2 + \frac{4GmR_s}{V_\infty^2}$$

Escape vel. at contact: $V_{esc}^2 = \frac{4GM}{R_s}$

Collisional cross section is thus:

$$\boxed{\sigma_{acc} = \pi R_s^2 \left(1 + \frac{V_{esc}^2}{V_\infty^2}\right)}$$

In a dynamically "cold" disk, $V_\infty \ll V_{esc}$

and thus, $\underline{\sigma_{acc} \gg \pi R_s^2}$

Thus, gravity enhances the collisional cross section. Called gravitational focusing

In slides, I provide more general form where the 2 masses are not necessarily equal

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For large embryo mass, $V_{esc} \gg V_\infty$

$\hookrightarrow F_g \gg 1 \Rightarrow F_g \simeq \left(\frac{V_{esc}^2}{V_\infty^2}\right) \sim \frac{2GM}{R V_\infty^2}$ ignoring smaller mass

Generally, V_∞ is indep of R

but $M \propto R^3$ for a given material density

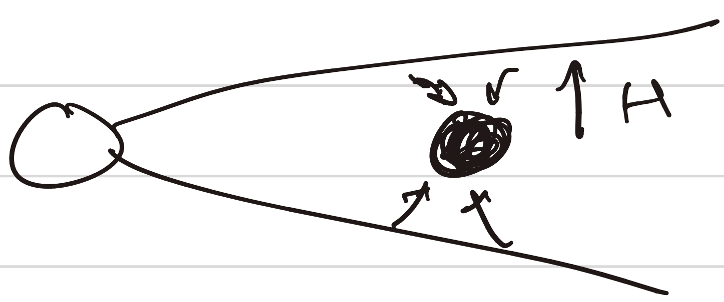
$\hookrightarrow F_g \propto \frac{R^3}{R} \propto R^2$

Thus, $\boxed{\frac{dm}{dt} \propto R^2 F_g \propto R^4 \Rightarrow \text{runaway growth}}$

Giant Planet Formation

- Characterize masses at different stages
- Consider surface escape speed: $V_{esc} = \sqrt{\frac{2GM_p}{R_s}}$

  radius of surface
- Planet embedded in disk will hold onto bound atmosphere if $V_{esc} > c_s$



gas wants to expand upward, but at surface (R_s) $V_{esc} > c_s$ guarantees <u>some</u> atmosphere

For a disk, $c_s = \left(\frac{H}{r}\right) v_K$

$$V_{esc} > c_s \Rightarrow M_p > \left(3/32\pi\right)^{1/2} \left(\frac{H}{r}\right)^3 \frac{M_*^{3/2}}{\rho_m^{1/2} a^{3/2}}$$

material density

- For icy body @ 5AU and $H/r = 0.05$, $M_p \gtrsim 5 \times 10^{-4} M_\oplus$

<u>Very small</u>

- But, there are stricter conditions
- Start with hydrostatic equilibrium with

$$\ddot{r} = -\frac{1}{\rho}\frac{dP}{dr} - \frac{GM_p}{r^2}$$

$M_t = M_p + M_{env}$
$= M_p + \varepsilon M_p$
$\approx M_p$ if $\varepsilon \ll 1$

HSE $\Rightarrow \ddot{r} = 0 \Rightarrow$ pressure gradient balances gravity

$$\frac{dP}{dr} = -\frac{GM_p}{r^2}\rho$$

- Let's make assumption of isothermal gas

$P = \rho c_s^2$ ← called equation of state.

Integrate HSE: $\ln\rho = \frac{GM_p}{c_s^2}\frac{1}{r} + \text{const.}$

Use boundary condition $\rho(r = r_{out}) = \rho_0$ (dens. of disk)

with

$$r_{out} = \frac{2GM_p}{c_s^2}$$ ← radius where $v_{esc} = c_s$

$$\rho(r) = \rho_0 e^{\left(\frac{GM_p}{c_s^2}\frac{1}{r} - \frac{1}{2}\right)}$$ ← envelope density profile in HSE

Since $\varepsilon \ll 1$, most of mass of envelope is in a thin shell above core

↳ $$M_{env} \simeq \frac{4}{3}\pi R_s^3 \rho(r = R_s)$$

Condition that $M_{env} > \varepsilon M_p$

↳ $$M_p \gtrsim \left(\frac{3}{4\pi\rho_m}\right)^{1/2}\left(\frac{c_s^2}{G}\right)^{3/2}\left[\ln\left(\frac{\varepsilon\rho_m}{\rho_0}\right)\right]^{3/2}$$

RHS $\propto c_s^3$ ⇒ M_p decreases with radius because disk is cooler in outer parts

Put in numbers

$r = 5\,\text{AU}$: $\rho_0 = 2\times10^{-11}\,\text{g/cm}^{-3}$, $c_s = 7\times10^4\,\text{cm/s}$, $\rho_m = 1\,\text{g/cm}^3$ ← icy

Assume $\varepsilon = 0.1$ ← appreciable but not dominant envelope

↳ $$M_p \gtrsim 0.2\,M_\oplus$$

$r = 1\,\text{AU}$: $\rho_0 = 6\times10^{-10}\,\text{g/cm}^3$, $c_s = 1.5\times10^5\,\text{cm/s}$

$\rho_m = 3\,\text{g/cm}^3$ ← rocky

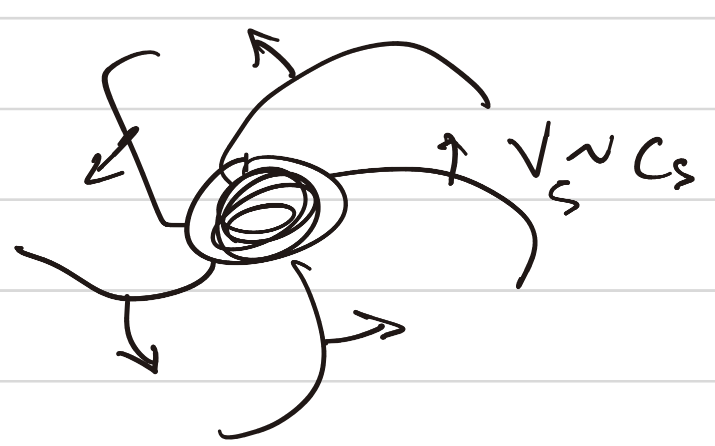
↳ $$M_p \gtrsim M_\oplus$$

- Thus, in inner solar sys., have to grow to large size before gas disappears, but smaller mass in outer disk.
- But what happens when $\varepsilon \sim 1$

$$\ddot{r} + \frac{1}{\rho}\frac{dP}{dr} = -\frac{G(M_p + M_{env})}{r^2}$$

much more complex, and no solution such that $\ddot{r} = 0 \Rightarrow$ no HSE possible!

- Consider disk where $Q \sim 1$
- Disk produces spiral structures with $v = v_s$



← these spiral structures can approach c_s and cause a shock in the gas, which heats gas

If $T\uparrow$, $c_s\uparrow$, $Q > 1$

$Q > 1 \Rightarrow$ no spirals, disk cools down $\Rightarrow c_s \downarrow$

But $c_s \downarrow \Rightarrow Q \downarrow \Rightarrow$ disk goes unstable again.

- Shocks don't always form, but turns out even for $v_s < c_s$, can get "gravitoturbulence" which also heats the gas

* Self-regulation if cooling is not too rapid

But fast cooling $\Rightarrow c_s \downarrow$ significantly and $Q < 1$ maintained.