Radiation Transfer and Radiation MHD in AthenaK

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Abstract

These notes develop the physical and numerical structure of radiation magnetohydrodynamics (radiation MHD) using the special relativistic transfer equation as the main pedagogical framework. The central ideas are (i) transport of the specific intensity, (ii) angular moments and radiation energy–momentum exchange, and (iii) the operator-split explicit transport and implicit source-term update used in a finite-solid-angle radiation module. A frame-by-frame algorithmic map makes explicit which part is solved in the local orthonormal tetrad and which part is solved in the fluid comoving frame. The final sections summarize the extension to general-relativistic MHD and curved-spacetime radiation transport.

Contents

1 Newtonian ideal MHD with radiation coupling

We first write the Newtonian ideal-MHD equations in Cartesian coordinates. Radiation enters as local energy and momentum source terms.

1.1 Mass conservation

$$\partial_t \rho + \partial_j \left(\rho v^j\right) = 0. \tag{1}$$

1.2 Momentum conservation

$$\partial_t \left(\rho v^i\right) + \partial_j \left[\rho v^i v^j + \left(P_\mathrm{g} + \frac{B^2}{2}\right) \delta^{ij} - B^i B^j\right] = -S^i_\mathrm{rad}. \tag{2}$$

The MHD momentum flux contains gas pressure, magnetic pressure, and magnetic tension. The quantity S^i_rad is the momentum gained by radiation per unit volume and time; the gas receives the opposite amount.

1.3 Energy conservation

Let e_g be the gas internal-energy density, with

$$P_\mathrm{g} = (\gamma - 1) e_\mathrm{g}. \tag{3}$$

The non-radiative MHD energy density is

$$E_\mathrm{MHD} = e_\mathrm{g} + \frac{1}{2}\rho v^2 + \frac{1}{2} B^2. \tag{4}$$

Its conservation equation is

$$\partial_t E_\mathrm{MHD} + \partial_j \left[\left(e_\mathrm{g} + P_\mathrm{g} + \frac{1}{2}\rho v^2 + B^2\right) v^j - (\boldsymbol{B}\cdot\boldsymbol{v}) B^j\right] = -S^E_\mathrm{rad}. \tag{5}$$

The flux combines enthalpy transport, kinetic-energy transport, magnetic energy transport, and the Poynting flux.

1.4 Induction equation

$$\partial_t B^i + \partial_j \left(v^j B^i - v^i B^j\right) = 0, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{B} = 0. \tag{6}$$

Equations (2) and (5) make the coupling strategy clear: after the radiation field changes, the opposite radiation energy and momentum increments are applied to the gas.

2 Radiation transfer equation

2.1 Specific intensity

The specific intensity

$$I_\nu = I_\nu(t, \boldsymbol{x}, \hat{\boldsymbol{n}}, \nu) \tag{7}$$

measures radiation energy carried through a surface, per unit time, area, solid angle, and frequency. In flat spacetime,

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \hat{\boldsymbol{n}}\cdot\boldsymbol{\nabla} I_\nu = \eta_\nu - \chi_\nu I_\nu. \tag{8}$$

The terms describe

- local time variation;
- free streaming along direction $\hat{\boldsymbol{n}}$;
- emission into the ray; and
- absorption or scattering out of the ray.

2.2 Fluid-frame emission, absorption, and scattering

An overbar denotes a quantity measured in the local fluid frame. For isotropic coherent scattering,

$$\bar{\eta}_{\bar{\nu}} = \bar{j}_{\bar{\nu}} + \bar{\chi}_{\mathrm{T},\bar{\nu}}\,\bar{J}_{\bar{\nu}}, \tag{9}$$

$$\bar{\chi}_{\bar{\nu}} = \bar{\chi}_{\mathrm{a},\bar{\nu}} + \bar{\chi}_{\mathrm{T},\bar{\nu}}, \tag{10}$$

$$\bar{J}_{\bar{\nu}} = \frac{1}{4\pi}\int \bar{I}_{\bar{\nu}}\,\mathrm{d}\bar{\Omega}. \tag{11}$$

Here $\bar{j}_{\bar{\nu}}$ is thermal emission, $\bar{\chi}_{\mathrm{a},\bar{\nu}}$ is true absorption, and $\bar{\chi}_{\mathrm{T},\bar{\nu}}$ is the scattering coefficient. Isotropic scattering removes radiation from one direction and re-emits the same energy into the angular mean $\bar{J}_{\bar{\nu}}$.

2.3 Lorentz transformation between lab and fluid frames

Define

$$A \equiv \frac{\bar{\nu}}{\nu} = \Gamma\left(1 - \hat{\boldsymbol{n}}\cdot\boldsymbol{\beta}\right), \qquad \boldsymbol{\beta} = \frac{\boldsymbol{v}}{c}, \qquad \Gamma = (1-\beta^2)^{-1/2}. \tag{12}$$

The standard invariants give

$$I_\nu = A^{-3}\bar{I}_{\bar{\nu}}, \qquad \eta_\nu = A^{-2}\bar{\eta}_{\bar{\nu}}, \qquad \chi_\nu = A\bar{\chi}_{\bar{\nu}}. \tag{13}$$

The frequency-dependent transfer equation can therefore be written with a fluid-frame interaction term while transport is performed in the lab frame.

2.4 Frequency-integrated, gray transfer equation

Define the frequency-integrated intensities

$$I \equiv \int_0^\infty I_\nu\,\mathrm{d}\nu, \qquad \bar{I} \equiv \int_0^\infty \bar{I}_{\bar{\nu}}\,\mathrm{d}\bar{\nu}. \tag{14}$$

Because $\mathrm{d}\nu = A^{-1}\,\mathrm{d}\bar{\nu}$,

$$I = A^{-4}\bar{I}. \tag{15}$$

For local thermodynamic equilibrium,

$$\int_0^\infty \bar{j}_{\bar{\nu}}\,\mathrm{d}\bar{\nu} = \bar{\chi}_\mathrm{P}\frac{c a_\mathrm{r} T_\mathrm{g}^4}{4\pi}. \tag{16}$$

A useful gray source decomposition is

$$\boxed{\frac{1}{c}\frac{\partial I}{\partial t} + \hat{\boldsymbol{n}}\cdot\boldsymbol{\nabla} I = A^{-3}\left[\bar{\chi}_\mathrm{P}\left(\frac{c a_\mathrm{r} T_\mathrm{g}^4}{4\pi} - \bar{J}\right) + \bar{\chi}_\mathrm{F}\left(\bar{J} - \bar{I}\right)\right],} \tag{17}$$

where

$$\bar{\chi}_\mathrm{F} \equiv \bar{\chi}_\mathrm{R} + \bar{\chi}_\mathrm{T}. \tag{18}$$

The two brackets have distinct physical roles:

- $\bar{\chi}_{\rm P}(ca_{\rm r}T_{\rm g}^4/4\pi - \bar{J})$ controls thermal emission and absorption, and therefore energy exchange;
- $\bar{\chi}_{\rm F}(\bar{J} - \bar{I})$ isotropizes the radiation field and therefore controls momentum exchange.

A gray Compton energy-exchange correction can schematically be added as

$$\bar{\chi}_{\rm T}\frac{4k_{\rm B}(T_{\rm g} - T_{\rm r})}{m_{\rm e}c^2}\,\bar{J}, \tag{19}$$

but it is omitted from the basic derivation below.

3 Angular moments: radiation energy and momentum

For the blackboard derivation, first consider the local fluid rest frame, $\boldsymbol{v} = 0$, so that $A = 1$ and barred and unbarred coordinates coincide. Define the angular moments

$$\bar{J} \equiv \frac{1}{4\pi}\int \bar{I}\,{\rm d}\bar{\Omega}, \tag{20}$$

$$\bar{H}^i \equiv \frac{1}{4\pi}\int \bar{I}\,\hat{\bar{n}}^i\,{\rm d}\bar{\Omega}, \tag{21}$$

$$\bar{K}^{ij} \equiv \frac{1}{4\pi}\int \bar{I}\,\hat{\bar{n}}^i\hat{\bar{n}}^j\,{\rm d}\bar{\Omega}. \tag{22}$$

3.1 Zeroth moment: radiation energy equation

Integrate equation (17) over solid angle. Since $\int \bar{I}\,{\rm d}\Omega/(4\pi) = \bar{J}$ and isotropic scattering conserves energy,

$$\boxed{\frac{1}{c}\frac{\partial \bar{J}}{\partial t} + \boldsymbol{\nabla}\cdot\bar{\boldsymbol{H}} = \bar{\chi}_{\rm P}\left(\frac{ca_{\rm r}T_{\rm g}^4}{4\pi} - \bar{J}\right).} \tag{23}$$

This is the radiation energy equation in moment form.

3.2 First moment: radiation momentum equation

Multiply equation (17) by $\hat{\bar{n}}^i$ and integrate. The isotropic emission terms vanish because $\int \hat{\bar{n}}^i\,{\rm d}\bar{\Omega} = 0$. The result is

$$\boxed{\frac{1}{c}\frac{\partial \bar{H}^i}{\partial t} + \partial_j\bar{K}^{ij} = -\bar{\chi}_{\rm F}\bar{H}^i.} \tag{24}$$

The flux is damped by absorption and scattering because both processes transfer radiation momentum to matter.

3.3 Physical radiation moments

The radiation energy density, flux, and pressure tensor are

$$E_{\rm r} = \frac{4\pi}{c}\bar{J}, \qquad F_{\rm r}^i = 4\pi\bar{H}^i, \qquad P_{\rm r}^{ij} = \frac{4\pi}{c}\bar{K}^{ij}. \tag{25}$$

Equations (23)–(24) become

$$\frac{\partial E_{\mathrm{r}}}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{F}_{\mathrm{r}} = c\bar{\chi}_{\mathrm{P}}\left(a_{\mathrm{r}}T_{\mathrm{g}}^4 - E_{\mathrm{r}}\right), \tag{26}$$

$$\frac{\partial}{\partial t}\left(\frac{F_{\mathrm{r}}^i}{c^2}\right) + \partial_j P_{\mathrm{r}}^{ij} = -\frac{\bar{\chi}_{\mathrm{F}}}{c}F_{\mathrm{r}}^i. \tag{27}$$

The variables E_{r} and $\boldsymbol{F}_{\mathrm{r}}/c^2$ are the radiation energy and momentum densities.

3.4 Energy–momentum exchange with the gas

The corresponding local source terms are

$$S_{\mathrm{rad}}^E = c\bar{\chi}_{\mathrm{P}}\left(a_{\mathrm{r}}T_{\mathrm{g}}^4 - E_{\mathrm{r}}\right), \qquad S_{\mathrm{rad}}^i = -\frac{\bar{\chi}_{\mathrm{F}}}{c}F_{\mathrm{r}}^i. \tag{28}$$

The gas receives the opposite source. Therefore, during a purely local interaction update,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(E_{\mathrm{g}} + E_{\mathrm{r}}\right) = 0, \tag{29}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho\boldsymbol{v} + \frac{\boldsymbol{F}_{\mathrm{r}}}{c^2}\right) = 0. \tag{30}$$

This equal-and-opposite exchange is the essential conservation property of the coupled algorithm.

3.5 Why the moment equations require a closure

The angular-moment equations form an infinite hierarchy. The zeroth-moment energy equation contains the first moment $\bar{\boldsymbol{H}}$, while the first-moment momentum equation contains the second moment $\bar{\mathsf{K}}$. If one writes an evolution equation for $\bar{\mathsf{K}}$, that equation contains a third angular moment, and so on. A finite moment method must therefore prescribe the highest retained moment in terms of lower moments. This prescription is called a *closure*.

A closure is not an additional conservation law. It is a model for the angular information that was discarded when $I(\hat{\boldsymbol{n}})$ was replaced by a small number of moments. Its validity consequently depends on the geometry of the radiation field.

3.5.1 Eddington closure

The simplest example is the Eddington closure,

$$\bar{K}^{ij} = \frac{1}{3}\bar{J}\,\delta^{ij}. \tag{31}$$

This relation is exact for an isotropic intensity distribution and is therefore appropriate in the optically thick diffusion limit. It cannot describe a strongly beamed radiation field, for which the pressure is highly anisotropic.

3.5.2 M1 closure

The M1 method evolves the radiation energy density and flux, or equivalently $\bar{J}$ and $\bar{\boldsymbol{H}}$, and closes the system with an algebraic variable Eddington tensor. Define the reduced flux

$$f \equiv \frac{|\bar{\boldsymbol{H}}|}{\bar{J}} = \frac{|\boldsymbol{F}_{\mathrm{r}}|}{cE_{\mathrm{r}}}, \qquad 0 \le f \le 1, \tag{32}$$

and, when $|\bar{\boldsymbol{H}}| \neq 0$, the unit flux direction

$$\hat{h}^i \equiv \frac{\bar{H}^i}{|\bar{\boldsymbol{H}}|}. \tag{33}$$

A commonly used M1 closure is [5]

$$\bar{K}^{ij} = \bar{J}\, D^{ij}, \tag{34}$$

$$D^{ij} = \frac{1-\xi}{2}\,\delta^{ij} + \frac{3\xi-1}{2}\,\hat{h}^i\hat{h}^j, \tag{35}$$

$$\xi(f) = \frac{3+4f^2}{5+2\sqrt{4-3f^2}}. \tag{36}$$

For $f \to 0$, one obtains $\xi \to 1/3$ and $D^{ij} \to \delta^{ij}/3$, recovering isotropic diffusion. For $f \to 1$, one obtains $\xi \to 1$ and $D^{ij} \to \hat{h}^i\hat{h}^j$, recovering a single freely streaming beam. M1 therefore provides a smooth interpolation between two important asymptotic limits.

M1 is useful because it is inexpensive, preserves a causal relation between energy and flux, and often works well when the radiation field is nearly isotropic or has one dominant propagation direction. It can consequently predict the bulk radiation energy and flux reasonably well in many problems, even though it does not reconstruct the full angular intensity.

The limitation is that $E_{\rm r}$ and $\boldsymbol{F}_{\rm r}$ do not uniquely determine the radiation pressure tensor. M1 effectively assumes that the angular distribution is axisymmetric about the local flux and has a single preferred lobe. Consider, for example, two equal beams propagating in the $+x$ and $-x$ directions. Their net flux vanishes,

$$\bar{\boldsymbol{H}} = 0, \tag{37}$$

but their true pressure tensor is

$$\bar{K}^{xx} = \bar{J}, \qquad \bar{K}^{yy} = \bar{K}^{zz} = 0. \tag{38}$$

Because M1 sees $f = 0$, it instead returns the isotropic result $\bar{K}^{ij} = \bar{J}\delta^{ij}/3$. Thus an isotropic radiation bath and two opposing beams have the same lower moments but very different second moments. No increase in spatial resolution can remove this representational ambiguity.

The same issue appears when beams cross, when several sources illuminate one point from different directions, or when sharp shadows form behind obstacles. M1 can merge crossing beams into one averaged direction, wash out shadows, or produce an inaccurate radiation force even when the local energy density looks reasonable. In this sense a closure can work for some aspects of a solution —for example the total energy or net flux—while failing for the angular structure and pressure tensor that control momentum feedback.

3.5.3 Why the finite-solid-angle method does not need an M1 closure

The AthenaK discrete-ordinates method evolves the angular intensities I_m directly. The moments are then calculated from the same angular distribution,

$$J = \sum_m w_m I_m, \qquad H^i = \sum_m w_m I_m \hat{n}^i_m, \qquad K^{ij} = \sum_m w_m I_m \hat{n}^i_m \hat{n}^j_m. \tag{39}$$

Therefore, no algebraic relation between J, $\boldsymbol{H}$, and K is needed. The tradeoff is computational cost and finite-angle artifacts such as ray effects. Unlike an algebraic closure, however, the angular approximation can be systematically improved by increasing the number of discrete solid-angle cells. This ability to represent multiple simultaneous beams is one of the main motivations for the finite-solid-angle method [6].

3.6 Diffusion limit

In an optically thick, nearly isotropic region, neglect $\partial_t \bar{H}^i$ in equation (24) and use the Eddington closure:

$$\frac{1}{3}\boldsymbol{\nabla}\bar{J} = -\bar{\chi}_{\mathrm{F}}\bar{\boldsymbol{H}}. \tag{40}$$

Thus

$$\boldsymbol{F}_{\mathrm{r}} = -\frac{c}{3\bar{\chi}_{\mathrm{F}}}\boldsymbol{\nabla}E_{\mathrm{r}}. \tag{41}$$

A flux-limited diffusion form replaces 1/3 by a limiter $\lambda(R)$,

$$\lambda\boldsymbol{\nabla}\bar{J} = -\bar{\chi}_{\mathrm{F}}\bar{\boldsymbol{H}}, \qquad R = \frac{|\boldsymbol{\nabla}\bar{J}|}{\bar{\chi}_{\mathrm{F}}\bar{J}}, \qquad \lambda(R) = \frac{2+R}{6+3R+R^2}. \tag{42}$$

4 Discrete ordinates and the AthenaK numerical framework

4.1 The numerical framework in one picture

The finite-solid-angle method evolves one specific intensity I_m for every spatial cell and discrete photon direction. The important organizing idea is that different pieces of the calculation are simplest in different local frames. The radiation transport variable and angular grid are defined in a local orthonormal tetrad, while emission, absorption, scattering, heating, cooling, and radiation force are evaluated in the fluid comoving frame [6].

geometry and fluid state $\longrightarrow$ tetrad-frame transport $\longrightarrow$ fluid-frame interaction $\longrightarrow$ conservative GRMHD update

The tetrad is a local orthonormal basis, so the photon momentum has the special-relativistic form

$$n^{\hat{\alpha}} = \left(1, \hat{n}^{\hat{1}}, \hat{n}^{\hat{2}}, \hat{n}^{\hat{3}}\right), \qquad \delta_{\hat{i}\hat{j}}\hat{n}^{\hat{i}}\hat{n}^{\hat{j}} = 1. \tag{43}$$

The conceptual algorithm is not tied to one unique tetrad construction. A HARM-like formulation for a stationary metric and a Valencia-style $3+1$ formulation for a time-dependent metric can each provide a suitable local orthonormal tetrad. What changes is the mapping between coordinate conserved variables and the tetrad, together with the geometric transport coefficients; the local special-relativistic transport and comoving interaction physics have the same structure. The published implementation summarized here specializes to a stationary spacetime and stationary tetrad [6]; a time-dependent metric requires the geometry and tetrad coefficients to be updated consistently in time.

4.2 One time-integration stage: step by step

At each Runge–Kutta or multistage time-integration stage, the update can be organized as follows.

Step 1: Build the local geometry and frames. Evaluate the metric, tetrad, photon directions, and the maps between coordinate, tetrad, and fluid frames. In a stationary spacetime most purely geometric coefficients can be precomputed.

Step 2: Reconstruct the tetrad-frame intensities. For every angular bin, reconstruct I_m from cell centers to spatial faces. In curved spacetime, also reconstruct to angular-cell faces, because geodesic bending advects intensity through angle space.

Step 3: Calculate explicit transport fluxes. Each angular bin behaves locally like a scalar advection equation. Upwind or Riemann-flux machinery supplies the physical-space fluxes; analogous upwind fluxes supply the angular-space transport generated by the geometry.

Step 4: Apply the transport flux divergences. Update the conserved radiation variables explicitly. This is the nonlocal part of the radiation update because neighboring spatial and angular cells exchange intensity.

Step 5: Transform to the fluid comoving frame. Lorentz-transform the post-transport intensities and photon directions from the tetrad to the instantaneous fluid rest frame.

Step 6: Solve the local matter–radiation source problem. Evaluate the comoving emissivities and opacities, and solve implicitly for emission, absorption, and scattering. This step determines both energy exchange—gas heating and cooling—and momentum exchange—the radiation force. It is cell-local but is usually the stiffest and most difficult part of the algorithm.

Step 7: Transform back and enforce four-momentum conservation. Transform the updated intensities to the tetrad, construct the radiation stress–energy tensor, and map it to the coordinate frame. Apply the equal-and-opposite change to the GRMHD conserved energy and momentum.

Step 8: Recover primitives and finish the stage. Recover fluid and radiation primitive variables, communicate ghost-zone data, perform mesh-refinement synchronization when needed, and impose boundary conditions before the next stage.

Part of equation	Natural frame	Numerical method	Physical role
Spatial and angular transport	Local tetrad	Explicit finite-volume upwind/Riemann flux	Photon propagation and geodesic bending
Matter–radiation source	Fluid comoving	Local implicit nonlinear solve	Heating/cooling and radiation force
Conservative coupling	Coordinate/GRMHD	Equal-and-opposite four-momentum update	Total energy–momentum conservation

4.3 Angular discretization

Replace the angular integrals by a finite set of directions $\hat{\boldsymbol{n}}_m$ and quadrature weights

$$w_m \equiv \frac{\Delta\Omega_m}{4\pi}, \qquad \sum_m w_m = 1. \tag{44}$$

A symmetric quadrature should satisfy

$$\sum_m w_m \hat{n}^i_m = 0, \tag{45}$$

$$\sum_m w_m (\hat{n}^x_m)^2 = \sum_m w_m (\hat{n}^y_m)^2 = \sum_m w_m (\hat{n}^z_m)^2 = \frac{1}{3}, \tag{46}$$

$$\sum_m w_m \hat{n}^i_m \hat{n}^j_m = 0, \qquad i \neq j. \tag{47}$$

The moments are then evaluated as

$$J = \sum_m w_m I_m, \qquad H^i = \sum_m w_m I_m \hat{n}^i_m, \qquad K^{ij} = \sum_m w_m I_m \hat{n}^i_m \hat{n}^j_m. \tag{48}$$

4.4 Angular aberration and transformed weights

For a moving fluid, the lab-frame direction $\hat{\boldsymbol{n}}$ and fluid-frame direction $\hat{\bar{\boldsymbol{n}}}$ are related by aberration:

$$\hat{\bar{\boldsymbol{n}}} = \frac{\hat{\boldsymbol{n}} - \Gamma\boldsymbol{\beta} + \dfrac{\Gamma^2}{\Gamma+1}(\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta})\boldsymbol{\beta}}{\Gamma(1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta})}. \tag{49}$$

Because $\mathrm{d}\bar{\Omega} = A^{-2}\,\mathrm{d}\Omega$, fluid-frame quadrature weights may be formed as

$$\bar{w}_m = \frac{A_m^{-2} w_m}{\sum_l A_l^{-2} w_l}, \tag{50}$$

where the normalization enforces $\sum_m \bar{w}_m = 1$.

4.5 Operator splitting

The radiation equation is split into a homogeneous transport problem and a local interaction problem. In flat spacetime the split is

$$\underbrace{\frac{1}{c}\frac{\partial I}{\partial t} + \hat{\boldsymbol{n}}\cdot\boldsymbol{\nabla} I = 0}_{\text{tetrad/lab-frame transport}}, \tag{51}$$

$$\underbrace{\frac{1}{c}\frac{\partial I}{\partial t} = A^{-3}\left[\bar{\chi}_{\mathrm{P}}\left(\frac{ca_{\mathrm{r}}T_{\mathrm{g}}^4}{4\pi} - \bar{J}\right) + \bar{\chi}_{\mathrm{F}}(\bar{J} - \bar{I})\right]}_{\text{fluid-frame matter–radiation interaction}}. \tag{52}$$

In GR the first operator also contains conservative transport through angle space, representing gravitational deflection and rotation of the tetrad basis. Schematically,

$$\boldsymbol{U}^n \xrightarrow{\text{explicit tetrad-frame transport}} \boldsymbol{U}^* \xrightarrow{\text{implicit fluid-frame coupling}} \boldsymbol{U}^{n+1}. \tag{53}$$

Transport is nonlocal, because it couples neighboring phase-space cells. The source solve is local to one spatial cell but couples all angular bins through radiation moments and the gas state.

4.6 Explicit transport in the tetrad frame

In a local orthonormal tetrad and in the absence of geometric angle transport, each discrete direction obeys the scalar advection equation

$$\frac{\partial I_m}{\partial \hat{t}} + c\partial_{\hat{j}}\left(\hat{n}^{\hat{j}}_m I_m\right) = 0. \tag{54}$$

Thus the advection step is conceptually straightforward: reconstruct left and right interface intensities, determine the upwind direction from the photon velocity, calculate a numerical flux with the same finite-volume/Riemann-flux architecture used elsewhere in AthenaK, and apply the flux divergence.

In curved spacetime, the coordinate flux direction is obtained by projecting $\hat{n}^{\hat{\alpha}}$ through the tetrad. Additional fluxes in angular space account for the change of photon direction along a null geodesic. Consequently, GR does not change the local scalar-advection character of the transport solve; it adds geometric coefficients and angle-space advection.

In optically thick moving media, a naive upwind flux with characteristic speed c can introduce excessive numerical diffusion. A useful decomposition is

$$I_m = \underbrace{\left(I_m - \frac{3\hat{\boldsymbol{n}}_m\cdot\boldsymbol{v}}{c}J\right)}_{\text{diffusive component}} + \underbrace{\frac{3\hat{\boldsymbol{n}}_m\cdot\boldsymbol{v}}{c}J}_{\text{advected component}}. \tag{55}$$

The two pieces can be transported with numerical fluxes chosen to preserve the correct diffusion and advection limits:

$$\frac{\partial I_m}{\partial t} + c\partial_j\left[\hat{n}^j_m\left(I_m - \frac{3\hat{\boldsymbol{n}}_m\cdot\boldsymbol{v}}{c}J\right)\right] + \partial_j\left[\hat{n}^j_m\left(3\hat{\boldsymbol{n}}_m\cdot\boldsymbol{v}\,J\right)\right] = 0. \tag{56}$$

This is an asymptotic-preserving idea: the numerical method should recover the correct physical diffusion limit without numerical diffusion proportional to $c\Delta x$ overwhelming the solution.

4.7 Why the interaction update is implicit

The characteristic interaction time is approximately

$$t_{\text{int}} \sim \frac{1}{c\chi}. \tag{57}$$

In optically thick cells, t_{int} can be much shorter than the MHD Courant timestep. An explicit update would then require $\Delta t \ll t_{\text{int}}$, so the local source terms are treated implicitly.

5 Implicit matter–radiation source update

This section gives a compact blackboard derivation of the gray source solve, which is the hardest part of the numerical framework. It is performed in the fluid comoving frame because thermal emission and absorption are isotropic there, and because heating/cooling and radiation force have their clearest physical interpretation there. During the basic local update, hold the density and velocity fixed and solve simultaneously for the new fluid-frame intensities and gas temperature. This blackboard derivation emphasizes the thermal-energy part of the source solve. In the full algorithm the anisotropic change of the intensities also changes radiation momentum; the corresponding radiation force is obtained from the change in the radiation stress–energy tensor and applied with the opposite sign to the fluid. In strongly radiation-dominated cells, the published method improves the fixed-velocity approximation with an estimated post-coupling velocity before enforcing exact total four-momentum conservation. Denote old and new states by superscripts $-$ and $+$.

For angular direction m, define

$$\alpha_m \equiv A_m c\Delta t, \qquad \bar{\chi}_{\text{F}} = \bar{\chi}_{\text{R}} + \bar{\chi}_{\text{T}}. \tag{58}$$

Using $I = A^{-4}\bar{I}$, the implicit source equation becomes

$$\bar{I}^+_m - \bar{I}^-_m = \alpha_m\left[\bar{\chi}_{\text{P}}\frac{ca_{\text{r}}}{4\pi}(T^+_{\text{g}})^4 + (\bar{\chi}_{\text{F}} - \bar{\chi}_{\text{P}})\bar{J}^+ - \bar{\chi}_{\text{F}}\bar{I}^+_m\right]. \tag{59}$$

Solving equation (59) for each intensity gives

$$\bar{I}_m^+ = \frac{\bar{I}_m^- + \alpha_m \bar{\chi}_{\rm P} \dfrac{c a_{\rm r}}{4\pi} (T_{\rm g}^+)^4 + \alpha_m (\bar{\chi}_{\rm F} - \bar{\chi}_{\rm P}) \bar{J}^+}{1 + \alpha_m \bar{\chi}_{\rm F}}. \tag{60}$$

Now impose

$$\bar{J}^+ = \sum_m \bar{w}_m \bar{I}_m^+. \tag{61}$$

Define

$$B_1 \equiv \sum_m \frac{\bar{w}_m \alpha_m}{1 + \alpha_m \bar{\chi}_{\rm F}}, \qquad B_2 \equiv \sum_m \frac{\bar{w}_m \bar{I}_m^-}{1 + \alpha_m \bar{\chi}_{\rm F}}. \tag{62}$$

Then

$$\boxed{\bar{J}^+ = \frac{B_2 + B_1 \bar{\chi}_{\rm P} \dfrac{c a_{\rm r}}{4\pi} (T_{\rm g}^+)^4}{1 - B_1 (\bar{\chi}_{\rm F} - \bar{\chi}_{\rm P})}.} \tag{63}$$

For an ideal gas,

$$e_{\rm g} = \frac{\rho \mathcal{R}}{\gamma - 1} T_{\rm g}. \tag{64}$$

Local energy conservation requires

$$\frac{\rho \mathcal{R}}{\gamma - 1} \left(T_{\rm g}^+ - T_{\rm g}^- \right) = -\frac{4\pi}{c} \left(\bar{J}^+ - \bar{J}^- \right). \tag{65}$$

Substitute equation (63) into (65). The result is a scalar nonlinear quartic equation for $T_{\rm g}^+$:

$$\frac{\rho \mathcal{R}}{\gamma - 1} \left(T_{\rm g}^+ - T_{\rm g}^- \right) + \frac{4\pi}{c} \left[\frac{B_2 + B_1 \bar{\chi}_{\rm P} \dfrac{c a_{\rm r}}{4\pi} (T_{\rm g}^+)^4}{1 - B_1 (\bar{\chi}_{\rm F} - \bar{\chi}_{\rm P})} - \bar{J}^- \right] = 0. \tag{66}$$

After solving equation (66) for a positive temperature, use equation (63) to obtain $\bar{J}^+$ and equation (60) to update every angular intensity.

5.1 Conservative coupling back to MHD

Transform the updated intensities back to the lab frame,

$$I_m^+ = A_m^{-4} \bar{I}_m^+, \tag{67}$$

and reconstruct the lab-frame radiation moments,

$$E_{\rm r}^+ = \frac{1}{c} \sum_m I_m^+ \Delta\Omega_m, \qquad F_{\rm r}^{i,+} = \sum_m I_m^+ \hat{n}_m^i \Delta\Omega_m. \tag{68}$$

The MHD conserved variables receive the opposite increments:

$$\Delta E_{\rm MHD} = -\left(E_{\rm r}^+ - E_{\rm r}^- \right), \tag{69}$$

$$\Delta(\rho v^i) = -\frac{1}{c^2} \left(F_{\rm r}^{i,+} - F_{\rm r}^{i,-} \right). \tag{70}$$

This is the discrete counterpart of total energy–momentum conservation.

5.2 Source-update summary

Within each spatial cell, the source part of the stage is therefore

$$\boxed{I^*_m \xrightarrow{\text{tetrad} \to \text{fluid boost}} \bar{I}^*_m \xrightarrow{\text{implicit heating/cooling and force solve}} \bar{I}^{n+1}_m \xrightarrow{\text{fluid} \to \text{tetrad boost}} I^{n+1}_m.} \tag{71}$$

The change in the radiation stress–energy tensor is then applied with the opposite sign to the GRMHD conserved variables. This completes Steps 5–7 of the stage-level algorithm given at the beginning of Section 4.

6 Extension to GRMHD

This section summarizes the GR extension. It can be presented briefly in the main lecture and retained in full as backup material.

6.1 Mass conservation

$$\nabla_\mu(\rho u^\mu) = 0, \tag{72}$$

or, in conservative coordinate form,

$$\partial_t \left(\sqrt{-g}\,\rho u^t\right) + \partial_j \left(\sqrt{-g}\,\rho u^j\right) = 0. \tag{73}$$

6.2 Stress–energy conservation

The ideal-GRMHD stress–energy tensor is

$$T^{\mu\nu}_{\rm MHD} = (\rho h + b^2)u^\mu u^\nu + \left(P_{\rm g} + \frac{b^2}{2}\right) g^{\mu\nu} - b^\mu b^\nu, \tag{74}$$

where

$$h = 1 + \frac{\gamma}{\gamma - 1}\frac{P_{\rm g}}{\rho}, \qquad b^2 = b_\mu b^\mu. \tag{75}$$

The conservation equation $\nabla_\mu T^\mu{}_\nu = G_\nu$ becomes

$$\partial_t \left(\sqrt{-g}\,T^t{}_\nu\right) + \partial_j \left(\sqrt{-g}\,T^j{}_\nu\right) = \sqrt{-g}\,T^\kappa{}_\lambda \Gamma^\lambda{}_{\nu\kappa} + \sqrt{-g}\,G_\nu. \tag{76}$$

The four-force G_ν is the energy–momentum transferred from radiation to matter. The Christoffel symbols are

$$\Gamma^\mu{}_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}\left(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta}\right). \tag{77}$$

6.3 Induction equation

The covariant ideal-MHD induction equation is

$$\nabla_\mu \left(u^\mu b^\nu - u^\nu b^\mu\right) = 0. \tag{78}$$

Its spatial components can be written

$$\partial_t \left[\sqrt{-g}\left(u^t b^i - u^i b^t\right)\right] + \partial_j \left[\sqrt{-g}\left(u^j b^i - u^i b^j\right)\right] = 0. \tag{79}$$

Defining

$$B^i \equiv u^t b^i - u^i b^t, \tag{80}$$

the time component supplies the divergence constraint

$$\partial_i \left(\sqrt{-g}\,B^i\right) = 0. \tag{81}$$

7 Radiation transfer in GR: the frame hierarchy

A practical GR radiation algorithm uses several frames because different parts of the problem are simplest in different representations.

(i) **Coordinate frame:** stores the conserved GRMHD variables, such as $\sqrt{-g}\,T^t{}_\nu$.

(ii) **Normal (Eulerian) frame:** naturally defines fluid primitive variables in a 3+1 decomposition; its timelike basis vector is normal to a constant-coordinate-time hypersurface.

(iii) **Local orthonormal tetrad frame:** represents photon directions and makes the local transfer physics look special relativistic.

(iv) **Fluid comoving frame:** evaluates emission, absorption, scattering, and the local source-term coupling.

The separation of roles is the key point: *transport is solved in the orthonormal tetrad, while matter–radiation coupling is solved in the fluid frame.* Coordinate and normal frames provide the interface to the chosen GRMHD formulation. The stationary-spacetime HARM-style equations used in the published method and a Valencia-style $3+1$ system use different coordinate conserved variables, but either can supply a local tetrad and receive the final four-momentum increment.

The conceptual sequence is

$$\begin{gathered}\text{coordinate conserved variables} \longleftrightarrow \text{normal-frame primitives} \\ \longleftrightarrow \text{orthonormal photon frame} \longleftrightarrow \text{fluid-frame source terms.}\end{gathered} \tag{82}$$

7.1 Photon directions in an orthonormal frame

Let (ζ, ψ) be polar and azimuthal angles in a local orthonormal tetrad. A null propagation direction can be represented as

$$\hat{n}^{\hat{t}} = 1, \qquad \hat{n}^{\hat{x}} = \sin\zeta\cos\psi, \qquad \hat{n}^{\hat{y}} = \sin\zeta\sin\psi, \qquad \hat{n}^{\hat{z}} = \cos\zeta. \tag{83}$$

Coordinate components follow from the tetrad:

$$\hat{n}^\mu = e^\mu{}_{\hat{\alpha}}\hat{n}^{\hat{\alpha}}. \tag{84}$$

7.2 Schematic conservative GR transfer equation

In curved spacetime, photons are advected through both physical space and angle space. A schematic conservative form is

$$\partial_t\left(\hat{n}^t\hat{n}_t I\right) + \frac{1}{\sqrt{-g}}\partial_j\left(\sqrt{-g}\,\hat{n}^j\hat{n}_t I\right) + \frac{1}{\sin\zeta}\partial_\zeta\left(\sin\zeta\,\dot{\zeta}\,\hat{n}_t I\right) + \partial_\psi\left(\dot{\psi}\,\hat{n}_t I\right) = \hat{n}_t\left(\hat{\eta} - \hat{\chi} I\right). \tag{85}$$

The physical-space flux transports photons across the grid. The angular fluxes $\dot{\zeta}$ and $\dot{\psi}$ represent gravitational bending and changes of the tetrad basis along a geodesic; they are constructed from Ricci rotation coefficients. The right-hand side is evaluated by transforming to the fluid frame.

The key pedagogical message is that GR adds geometry, redshift, and angular advection, but the local interaction physics remains the same energy–momentum exchange developed in Sections 2–5.

For the finite-solid-angle scheme, the homogeneous terms on the left-hand side are handled explicitly as flux divergences in physical and angular space. The right-hand side is operator-split, Lorentz-transformed to the fluid frame, and handled with the locally implicit source solve. Afterward the intensity is transformed back, and the change in radiation four-momentum is subtracted from the fluid four-momentum. This is the GR version of the same transport–source– conservation cycle developed in the non-GR sections.

8 Summary

This lecture can be summarized by the following chain:

$$\begin{gathered} I(t,\boldsymbol{x},\hat{\boldsymbol{n}}) \longrightarrow (J,\boldsymbol{H},\mathsf{K}) \longrightarrow (E_{\rm r},\boldsymbol{F}_{\rm r},\mathsf{P}_{\rm r}) \\ \longrightarrow \text{radiation energy–momentum exchange} \longrightarrow \text{conservative coupling to MHD.} \end{gathered} \tag{86}$$

The numerical architecture mirrors the physics:

$$\boxed{\begin{gathered}\text{explicit transport across cells} + \text{implicit local source coupling} \\ +\text{equal-and-opposite MHD update}\end{gathered}} \tag{87}$$

The non-GR derivation makes the physical content transparent. The GR formulation preserves the same structure while adding covariant conservation, frame transformations, gravitational redshift, and angular advection caused by geodesic bending.

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