Angular Momentum Transport and Energy Radiation in Accretion Discs:
Viscosity, Turbulence, Winds, Poynting Flux, and Non-ideal MHD Heating

Lecture notes for AthenaK summer school. Zhaohuan Zhu aided by Claude.

Conventions. Units absorb the 4π into $\mathbf{B}$ (as in Athena++), so the Maxwell stress is $-B_R B_\phi$, the current density is $\mathbf{J} = \nabla \times \mathbf{B}$, and Ohmic heating is $\eta_{\rm O} J^2$; restore factors of 4π and c for Gaussian units. Cylindrical coordinates (R, ϕ, z); spherical-polar (r, θ, ϕ) in §8. $\langle\cdot\rangle$ denotes an azimuthal (and, in practice, time) average. Subscript p denotes poloidal components: $\mathbf{v} = \mathbf{v}_p + v_\phi \hat{\boldsymbol{\phi}}$, $\mathbf{B} = \mathbf{B}_p + B_\phi \hat{\boldsymbol{\phi}}$.

1 Starting equations

Compressible non-ideal MHD with an explicit molecular (Navier–Stokes) viscosity:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{1}$$

$$\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot \Big[\rho \mathbf{v}\mathbf{v} + P\,\mathsf{I} + \Pi\Big] = \mathbf{J} \times \mathbf{B} - \rho \nabla \Phi, \qquad \Phi = -\frac{GM}{r}, \tag{2}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}, \qquad \mathbf{E} = -\mathbf{v} \times \mathbf{B} + \eta_{\rm O} \mathbf{J} + \eta_{\rm H}\, \mathbf{J} \times \hat{\mathbf{b}} + \eta_{\rm A}\, \mathbf{J}_\perp, \tag{3}$$

where $\hat{\mathbf{b}} = \mathbf{B}/|\mathbf{B}|$, $\mathbf{J}_\perp = -(\mathbf{J} \times \hat{\mathbf{b}}) \times \hat{\mathbf{b}}$ is the current component perpendicular to $\mathbf{B}$, and $(\eta_{\rm O}, \eta_{\rm H}, \eta_{\rm A})$ are the Ohmic, Hall, and ambipolar diffusivities. In these notes Ohm's law (3) is taken as given; the three-term structure, the microphysical coefficients, and the reason each term looks the way it does are *derived* from multifluid force balance in Appendix A — assign it as reading before §5. The viscous stress tensor is

$$\Pi_{ij} = -\rho \nu \Big(\partial_i v_j + \partial_j v_i - \tfrac{2}{3} \delta_{ij}\, \nabla \cdot \mathbf{v}\Big). \tag{4}$$

For a differentially rotating flow $\mathbf{v} = R\,\Omega(R, z)\,\hat{\boldsymbol{\phi}}$ the only surviving off-diagonal components are

$$\boxed{\Pi_{R\phi} = -\rho \nu\, R \frac{\partial \Omega}{\partial R}, \qquad \Pi_{\phi z} = -\rho \nu\, R \frac{\partial \Omega}{\partial z}.} \tag{5}$$

For Keplerian rotation $R\,\partial_R \Omega = -\frac{3}{2}\Omega$, so $\Pi_{R\phi} = +\frac{3}{2}\rho\nu\Omega > 0$: viscosity transports angular momentum *outward* automatically. Everything downstream inherits this sign.

2 Why molecular viscosity is not enough

Kinetic theory: $\nu_{\rm mol} \simeq \frac{1}{3} \lambda_{\rm mfp}\, c_s$ with $\lambda_{\rm mfp} = 1/(n \sigma_{\rm H_2})$ and $\sigma_{\rm H_2} \approx 2 \times 10^{-15}\,{\rm cm}^2$. At $R = 1$ au in a protoplanetary disc: $n \sim 10^{14}\,{\rm cm}^{-3}$, $T \sim 300$ K so $c_s \sim 10^5\,{\rm cm\,s}^{-1}$, $H \sim 0.04R \sim 6 \times 10^{11}$ cm.

Then

$$\lambda_{\rm mfp} \sim \frac{1}{10^{14}\cdot 2\times 10^{-15}} \sim 5\ \text{cm}, \qquad \nu_{\rm mol} \sim 2\times 10^{5}\ \text{cm}^2\,\text{s}^{-1}. \tag{6}$$

Three equivalent statements of the failure:

$$t_\nu = \frac{R^2}{\nu_{\rm mol}} \sim \frac{(1.5\times 10^{13}\,\text{cm})^2}{2\times 10^5\,\text{cm}^2\,\text{s}^{-1}} \sim 3\times 10^{13}\ \text{yr} \quad \text{vs. observed } t_{\rm disc}\sim 3\times 10^6\ \text{yr}; \tag{7}$$

$$\alpha_{\rm mol} \equiv \frac{\nu_{\rm mol}}{c_s H} \sim \frac{\lambda_{\rm mfp}}{3H} \sim 3\times 10^{-12} \quad \text{vs. required } \alpha \sim 10^{-3}\text{–}10^{-2}; \tag{8}$$

$$\text{Re} = \frac{c_s H}{\nu_{\rm mol}} \sim \frac{3H}{\lambda_{\rm mfp}} \sim 10^{11}. \tag{9}$$

Molecular transport falls short by $\sim$9 orders of magnitude, while the enormous Reynolds number says the flow *wants* to be turbulent if anything can destabilize it. The subtlety: a Keplerian disc is **Rayleigh stable**, since the specific angular momentum $\ell = \sqrt{GMR}$ increases outward,

$$\kappa^2 = \frac{1}{R^3}\frac{\mathrm{d}\ell^2}{\mathrm{d}R} = \Omega^2 > 0, \tag{10}$$

so axisymmetric hydrodynamic perturbations are linearly stable despite the huge Re. This is why a weak magnetic field is transformative: the magnetorotational instability (MRI; Balbus & Hawley 1991, 1998) replaces the Rayleigh criterion $\mathrm{d}\ell^2/\mathrm{d}R < 0$ with $\mathrm{d}\Omega^2/\mathrm{d}R < 0$, which Keplerian discs satisfy everywhere, with maximal growth rate $\frac{3}{4}\Omega$. Turbulence is therefore not "enhanced viscosity" by assumption — whether it *acts* like one is exactly what the averaged equations below let us test.

3 Reynolds decomposition: exact averaged angular momentum equation

Write $v_\phi = v_K(R) + \delta v_\phi$. The exact conservation law for the z-component of angular momentum, $\rho R v_\phi$, in cylindrical coordinates is

$$\frac{\partial(\rho R v_\phi)}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}\Big[R\cdot R\big(\rho v_R v_\phi - B_R B_\phi + \Pi_{R\phi}\big)\Big] + \frac{1}{R}\frac{\partial}{\partial \phi}\big[\cdots\big] + \frac{\partial}{\partial z}\Big[R\big(\rho v_z v_\phi - B_z B_\phi + \Pi_{\phi z}\big)\Big] = 0. \tag{11}$$

(The non-ideal terms do not enter here: Ohm's law modifies the induction equation, not the momentum flux; the Lorentz force is $\mathbf{J}\times\mathbf{B}$ regardless of η.) Azimuthal averaging kills the ∂_ϕ term. Substituting the decomposition and using continuity to remove the mean Keplerian part, $\partial_t(\rho R v_K) + \nabla\cdot(\rho \mathbf{v}\, R v_K) = \rho v_R\, \partial_R(R v_K)$, what survives is *exact* — no closure has been invoked:

$$\boxed{\frac{\partial\langle \rho R\, \delta v_\phi\rangle}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}\Big(R^2 \langle T_{R\phi}\rangle\Big) + R\,\frac{\partial \langle T_{\phi z}\rangle}{\partial z} + \langle \rho v_R\rangle \frac{\partial (R v_K)}{\partial R} = 0,} \tag{12}$$

with the three-part total stresses

$$T_{R\phi} = \underbrace{\langle \rho\, \delta v_R\, \delta v_\phi\rangle}_{\text{Reynolds}} \underbrace{- \langle B_R B_\phi\rangle}_{\text{Maxwell}} \underbrace{- \rho\nu R\frac{\partial\Omega}{\partial R}}_{\text{viscous}}, \tag{13}$$

$$T_{\phi z} = \langle \rho\, \delta v_z\, \delta v_\phi\rangle - \langle B_z B_\phi\rangle - \rho\nu R\frac{\partial\Omega}{\partial z}. \tag{14}$$

This is Eq. (8) of Zhu, Jiang & Stone (2020) with the dissipation tensor σ written out explicitly.

Remark 1 (correlations, not randomness). Turbulence contributes only through *correlations*: transport requires δv_R and δv_ϕ to be correlated, which isotropic random motion would not supply. The MRI produces $\langle \delta v_R\, \delta v_\phi \rangle > 0$ and $\langle B_R B_\phi \rangle < 0$ dynamically, with the Maxwell part exceeding the Reynolds part by a factor $\sim$3–4 (Hawley, Gammie & Balbus 1995); the simulations of Zhu & Stone (2018, their Fig. 11) reproduce this ratio.

Remark 2 (α is a closure). The Shakura–Sunyaev (1973) prescription is precisely the closure $T_{R\phi} = \alpha P$: the *assumption* that the turbulent correlation behaves like an enhanced $\Pi_{R\phi}$ with eddy viscosity

$$\nu_{\rm turb} = \tfrac{2}{3}\,\alpha\, c_s H \qquad \big(\text{dimensionally } \nu_{\rm turb} \sim \delta v\, \ell_{\rm eddy} \sim \alpha c_s H\big). \tag{15}$$

All three stresses enter the angular momentum equation *identically*; the physics differs only in where each dissipates (§5–§6).

Remark 3 (what δ means at a wind surface — excess flux). The δ's are deviations from the mean Keplerian state, *not* necessarily turbulence: at a laminar wind base $\delta v_z = v_z$ (the mean interior $v_z \simeq 0$) and $\delta v_\phi = v_\phi - v_K$. Note also what the continuity step above quietly did to the vertical term: the advection of *Keplerian* angular momentum by the wind mass flux, $\rho v_z \cdot R v_K$, was absorbed into the R-dependence of $\dot M$ (mass loss makes $\partial \dot M/\partial R \neq 0$), leaving in $T_{\phi z}$ only the *excess* angular-momentum flux. At a wind surface, anticipating the invariant l of §7 ($v_\phi - B_\phi/k = l/R$),

$$T_{\phi z}\big|_{\rm base} = \rho v_z\,(v_\phi - v_K) - B_z B_\phi = \rho v_z\,\frac{l-j}{R}, \qquad j \equiv R v_K. \tag{16}$$

The physics of the split: gas that leaves takes its own j with it *for free* — removing a parcel along with the angular momentum it already owned exerts no torque on the disc that remains; only the excess $l - j$, paid out to the parcel by the field, brakes the disc and drives accretion. This is why $(\lambda - 1)$, not λ, will appear in the wind-driven accretion rate of §4.2. (In Bai 2026 the same split is explicit rather than implicit: his Eq. 4 carries a separate mass-loss term $2\pi R \dot\Sigma_w\, j$ alongside $T_{z\phi}$, and his Eq. 11 is literally $\dot M\, {\rm d}j/{\rm d}R = (\partial \dot M_{\rm wind}/\partial R)(l - j)$.)

4 Height integration: internal stress vs. surface stress

Integrate Eq. (12) over z between the disc surfaces $z_\pm$, assume steady state, and define

$$\dot M \equiv -2\pi R \int \langle \rho v_R \rangle\, {\rm d}z \quad \text{(positive for inflow)}, \qquad W_{R\phi} \equiv \int \langle T_{R\phi} \rangle\, {\rm d}z. \tag{17}$$

Then

$$\boxed{\frac{\dot M}{2\pi}\frac{\partial (R v_K)}{\partial R} = \frac{\partial}{\partial R}\Big(R^2 W_{R\phi}\Big) + R^2 \Big[\langle T_{\phi z} \rangle\Big]_{z_-}^{z_+}.} \tag{18}$$

Everything about "what drives accretion" is in this line: the divergence of the internal R–ϕ torque, plus the ϕ–z stress evaluated *at the surfaces* — the wind torque. Note that $T_{\phi z}$ *inside* the disc dropped out of the integral: internal vertical stress only redistributes angular momentum between layers. It drives the vertically sheared "surface accretion / midplane outflow" pattern of Zhu & Stone (2018) but cannot change $\dot M$.

4.1 Viscous/turbulent limit (no wind)

Drop the surface term, use $\partial_R(Rv_K) = v_K/2$, and integrate radially with the zero-torque inner boundary condition $R^2 W_{R\phi} \to \dot{M} R_{\rm in} v_{K,\rm in}/2\pi$ at $R_{\rm in}$:

$$W_{R\phi} = \frac{\dot{M} v_K}{2\pi R}\left(1 - \sqrt{\frac{R_{\rm in}}{R}}\right). \tag{19}$$

If the stress is viscous (molecular or eddy), $W_{R\phi} = \frac{3}{2}\nu\Sigma\Omega$, giving the Lynden-Bell & Pringle (1974) relation

$$\boxed{\nu\Sigma = \frac{\dot{M}}{3\pi}\left(1 - \sqrt{\frac{R_{\rm in}}{R}}\right).} \tag{20}$$

4.2 Wind-driven limit (result; derivation in §7)

The ϕ–z stress at the wind base will be shown in §7 to collapse to (using Remark 3 and the invariant l of §7)

$$T_{\phi z}\big|_{\rm base} = \rho v_z \frac{l - j}{R} = \rho v_z\,(\lambda - 1)\,v_K, \qquad l = \omega R_A^2, \quad j = R v_K, \quad \lambda \equiv \left(\frac{R_A}{R}\right)^2 : \tag{21}$$

each gram of wind *carries away* the specific angular momentum $l = \omega R_A^2$ of its Alfvén radius, not of its footpoint — but the j it already owned leaves with it for free (Remark 3), so only the excess $(\lambda - 1)j$ torques the disc left behind. Balancing against Eq. (18) gives the wind-driven accretion rate

$$\dot{M}_{\rm acc,wind} \simeq 2(\lambda - 1)\,R\,\frac{\partial \dot{M}_{\rm loss}}{\partial R}. \tag{22}$$

Fiducial numbers from Zhu & Stone (2018): $\lambda \sim 10$, $\rho v_z \sim 2 \times 10^{-6}$ give $\dot{M}_{\rm acc,wind} \sim 3 \times 10^{-4}$ against $\dot{M} \sim 5 \times 10^{-3}$ — the wind supplies only ∼5% of the torque (rising to ∼20% in the thicker FU Ori disc of the 2020 paper).

5 The energy equations, derived step by step

We now build the energy budget channel by channel. The point of doing it this way is that the conversion terms appear in *pairs with opposite signs* in adjacent channels, so one can trace exactly how orbital energy becomes heat.

5.1 Kinetic energy

Dot the momentum equation with $\mathbf{v}$. Using $\mathbf{v} \cdot \nabla \cdot (\rho\mathbf{v}\mathbf{v}) = \nabla \cdot (\frac{1}{2}\rho v^2 \mathbf{v})$ (with continuity) and $\mathbf{v} \cdot (\nabla \cdot \Pi) = \nabla \cdot (\Pi \cdot \mathbf{v}) - \Pi_{ij}\partial_j v_i$:

$$\boxed{\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho v^2\right) + \nabla \cdot \left[\tfrac{1}{2}\rho v^2 \mathbf{v} + P\mathbf{v} + \Pi \cdot \mathbf{v}\right] = P\,\nabla \cdot \mathbf{v} \underbrace{-\,Q_{\rm visc}}_{\to\ \text{heat}} \underbrace{+\,\mathbf{v} \cdot (\mathbf{J} \times \mathbf{B})}_{\leftarrow\ \text{magnetic}} - \rho\,\mathbf{v} \cdot \nabla\Phi,} \tag{23}$$

where the viscous dissipation function is

$$Q_{\rm visc} = -\Pi_{ij}\,\partial_j v_i = 2\rho\nu\,e_{ij}e_{ij} - \tfrac{2}{3}\rho\nu(\nabla \cdot \mathbf{v})^2 \;\geq 0, \qquad e_{ij} = \tfrac{1}{2}(\partial_i v_j + \partial_j v_i). \tag{24}$$

For pure Keplerian shear,

$$\boxed{Q_{\rm visc} = \rho\nu\Big(R\frac{\partial\Omega}{\partial R}\Big)^2 = \frac{9}{4}\,\rho\nu\,\Omega^2.} \tag{25}$$

Sign conventions to note on the board: $-Q_{\rm visc}$ removes kinetic energy (it will reappear with a $+$ sign in the internal energy equation); $+\mathbf{v}\cdot(\mathbf{J}\times\mathbf{B})$ is the work done *by* the Lorentz force on the gas (it will reappear with a $-$ sign in the magnetic energy equation).

5.2 Magnetic energy and the Poynting flux $\mathbf{E}\times\mathbf{B}$, in full

(a) Where the Poynting flux comes from. Dot the induction equation with $\mathbf{B}$ and use the identity $\nabla\cdot(\mathbf{E}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{E}) - \mathbf{E}\cdot(\nabla\times\mathbf{B})$:

$$\frac{\partial}{\partial t}\Big(\frac{B^2}{2}\Big) = -\mathbf{B}\cdot(\nabla\times\mathbf{E}) = -\nabla\cdot(\mathbf{E}\times\mathbf{B}) - \mathbf{E}\cdot\mathbf{J}. \tag{26}$$

The flux of magnetic energy is therefore $\mathbf{S}\equiv\mathbf{E}\times\mathbf{B}$ — the Poynting vector (in units with c, 4π absorbed) — *whatever* Ohm's law is; the physics of the plasma enters only through what $\mathbf{E}$ is.

(b) $\mathbf{E}\cdot\mathbf{J}$ term by term. Insert the generalized Ohm's law (3):

$$\begin{aligned}
(-\mathbf{v}\times\mathbf{B})\cdot\mathbf{J} &= +\,\mathbf{v}\cdot(\mathbf{J}\times\mathbf{B}) && \text{(cyclic permutation: transfer to kinetic)}\\
(\eta_{\rm O}\mathbf{J})\cdot\mathbf{J} &= \eta_{\rm O}J^2 \;\geq 0 && \textbf{(Ohmic heating)}\\
(\eta_{\rm H}\,\mathbf{J}\times\hat{\mathbf{b}})\cdot\mathbf{J} &= 0 && \textbf{(Hall does no work)}\\
(\eta_{\rm A}\,\mathbf{J}_\perp)\cdot\mathbf{J} &= \eta_{\rm A}J_\perp^2 \;\geq 0 && \textbf{(ambipolar heating)},
\end{aligned} \tag{27}$$

so

$$\boxed{\frac{\partial}{\partial t}\Big(\frac{B^2}{2}\Big) + \nabla\cdot(\mathbf{E}\times\mathbf{B}) = \underbrace{-\,\mathbf{v}\cdot(\mathbf{J}\times\mathbf{B})}_{\leftarrow\ \text{kinetic}}\ \underbrace{-\,\eta_{\rm O}J^2 - \eta_{\rm A}J_\perp^2}_{\rightarrow\ \text{heat}}.} \tag{28}$$

(c) The Poynting flux itself, term by term. Substitute Ohm's law into $\mathbf{S}=\mathbf{E}\times\mathbf{B}$ and evaluate each cross product with $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \mathbf{b}\,(\mathbf{a}\cdot\mathbf{c}) - \mathbf{a}\,(\mathbf{b}\cdot\mathbf{c})$:

$$\begin{aligned}
(-\mathbf{v}\times\mathbf{B})\times\mathbf{B} &= B^2\mathbf{v} - (\mathbf{v}\cdot\mathbf{B})\,\mathbf{B} && \text{(ideal)}\\
(\eta_{\rm O}\mathbf{J})\times\mathbf{B} &= \eta_{\rm O}\,\mathbf{J}\times\mathbf{B} && \text{(Ohmic flux)}\\
(\eta_{\rm H}\,\mathbf{J}\times\hat{\mathbf{b}})\times\mathbf{B} &= B\big[\hat{\mathbf{b}}(\mathbf{J}\cdot\hat{\mathbf{b}}) - \mathbf{J}\big] = -\,B\,\mathbf{J}_\perp && \text{(Hall flux)}\\
(\eta_{\rm A}\,\mathbf{J}_\perp)\times\mathbf{B} &= \eta_{\rm A}\,\mathbf{J}_\perp\times\mathbf{B} && \text{(ambipolar flux)},
\end{aligned} \tag{29}$$

hence the complete Poynting flux:

$$\boxed{\mathbf{S} = \mathbf{E}\times\mathbf{B} = \underbrace{B^2\mathbf{v} - (\mathbf{v}\cdot\mathbf{B})\,\mathbf{B}}_{\text{ideal}} + \eta_{\rm O}\,\mathbf{J}\times\mathbf{B} \;-\; \eta_{\rm H}\,B\,\mathbf{J}_\perp \;+\; \eta_{\rm A}\,\mathbf{J}_\perp\times\mathbf{B}.} \tag{30}$$

Remarks:

- **Ideal part decomposed.** Write $B^2\mathbf{v}-(\mathbf{v}\cdot\mathbf{B})\mathbf{B} = \big[\frac{B^2}{2}\mathbf{v}\big] + \big[\frac{B^2}{2}\mathbf{v} - (\mathbf{v}\cdot\mathbf{B})\mathbf{B}\big]$: the first bracket is *advection* of magnetic energy; the second is the rate of work done by magnetic pressure and tension (equivalently $-\mathbf{v}\cdot\mathsf{M}$ with the Maxwell stress tensor $\mathsf{M}_{ij} = B_iB_j - \frac{1}{2}B^2\delta_{ij}$).

- **Disc geometry, leading order.** With $v_\phi \simeq v_K$ dominant, $S_R = v_R B^2 - B_R(\mathbf{v}\cdot\mathbf{B}) \simeq -\,v_K B_R B_\phi$ and $S_z \simeq -\,v_K B_z B_\phi$: *Maxwell stress times* v_K — torque times angular velocity, exactly like a driveshaft. This is why magnetic energy extracted at one height can emerge as radiation at another: the Poynting flux is a transport channel the viscous model does not possess.

- **Wind geometry.** For the steady axisymmetric wind of §7, with $\mathbf{v}_p \parallel \mathbf{B}_p$ the ideal electric field becomes $\mathbf{E} = -\big[v_\phi - (k/\rho)B_\phi\big]\hat{\boldsymbol{\phi}} \times \mathbf{B}_p = -\,\omega R\,\hat{\boldsymbol{\phi}} \times \mathbf{B}_p$ (the invariants k, ω are derived in §7). Then, using $(\hat{\boldsymbol{\phi}} \times \mathbf{B}_p) \times \mathbf{B}_p = -B_p^2\hat{\boldsymbol{\phi}}$ and $(\hat{\boldsymbol{\phi}} \times \mathbf{B}_p) \times B_\phi\hat{\boldsymbol{\phi}} = B_\phi\mathbf{B}_p$,

$$\boxed{\mathbf{S} = \mathbf{E}\times\mathbf{B} = \omega R\,B_p^2\,\hat{\boldsymbol{\phi}} \;-\; \omega R\,B_\phi\,\mathbf{B}_p, \qquad \mathbf{S}_p = -\,\omega R\,B_\phi\,\mathbf{B}_p.} \tag{31}$$

  The poloidal Poynting flux along the wind is (field-line rotation rate) $\times$ (magnetic torque): this is the term that will build the Bernoulli invariant in §7.

- **Hall paradox resolved.** The Hall flux $-\eta_{\rm H} B\,\mathbf{J}_\perp$ is generally nonzero even though $\mathbf{E}_{\rm H}\cdot\mathbf{J} = 0$: the Hall effect *transports* magnetic energy (physically, via whistler waves) without dissipating any — dispersive, not dissipative. It also famously breaks the $\pm\mathbf{B}$ symmetry of the MRI. (Appendix A.6 shows *why* from the multifluid picture: $\mathbf{E}'\cdot\mathbf{J}$ equals the total interspecies friction, and the Hall field drives a frictionless $\mathbf{E}\times\mathbf{B}$ drift.)

- **Ambipolar.** Microscopically the heating is frictional: ion–neutral drift $\Delta\mathbf{v}_{\rm in} = (\mathbf{J}\times\mathbf{B})/(\gamma_i\rho_i\rho_n)$ dissipates $Q_{\rm AD} = \gamma_i\rho_i\rho_n|\Delta\mathbf{v}_{\rm in}|^2 = \eta_{\rm A} J_\perp^2$, with $\eta_{\rm A} = B^2/(\gamma_i\rho_i\rho)$ (derived in Appendix A.5).

- **Ohmic.** Heating $\eta_{\rm O} J^2$ concentrates in current sheets, which MRI turbulence produces copiously at small scales; the split between viscous and resistive dissipation of the cascade depends on the magnetic Prandtl number $\mathrm{Pm} = \nu/\eta_{\rm O}$, with resistive termination dominating for $\mathrm{Pm} \lesssim 1$.

5.3 Internal energy, the full total energy equation, and the bookkeeping theorem

The internal energy $E_g = P/(\gamma-1)$ obeys

$$\boxed{\frac{\partial E_g}{\partial t} + \nabla\cdot(E_g\mathbf{v}) + P\,\nabla\cdot\mathbf{v} = \; Q_{\rm visc} \;+\; \eta_{\rm O} J^2 \;+\; \eta_{\rm A} J_\perp^2 \;-\; \nabla\cdot\mathbf{F}_{\rm rad}.} \tag{32}$$

Now sum the three channels, Eqs. (23), (28) and (32). The pairs $\pm Q_{\rm visc}$, $\pm\mathbf{v}\cdot(\mathbf{J}\times\mathbf{B})$, $\pm\eta_{\rm O}J^2$, $\pm\eta_{\rm A}J_\perp^2$, and $\pm P\nabla\cdot\mathbf{v}$ all cancel, and the flux $E_g\mathbf{v} + P\mathbf{v}$ combines into the enthalpy flux $\frac{\gamma}{\gamma-1}P\mathbf{v}$. The result, written out with *every* term explicit and labeled, is the full total energy equation:

$$\boxed{\begin{aligned}\frac{\partial}{\partial t}\,\underbrace{\left(\frac{P}{\gamma-1} + \frac{1}{2}\rho v^2 + \frac{B^2}{2}\right)}_{E\,=\,\text{internal + kinetic + magnetic}} &+ \nabla\cdot\Big[\underbrace{\frac{\gamma}{\gamma-1}P\,\mathbf{v}}_{\text{enthalpy}} + \underbrace{\frac{1}{2}\rho v^2\,\mathbf{v}}_{\text{KE advection}} + \underbrace{B^2\mathbf{v} - (\mathbf{v}\cdot\mathbf{B})\,\mathbf{B}}_{\text{ideal Poynting}} + \underbrace{\Pi\cdot\mathbf{v}}_{\text{viscous work}} \\ &+ \underbrace{\eta_{\rm O}\,\mathbf{J}\times\mathbf{B}}_{\text{Ohmic Poynting}} - \underbrace{\eta_{\rm H}\,B\,\mathbf{J}_\perp}_{\text{Hall Poynting}} + \underbrace{\eta_{\rm A}\,\mathbf{J}_\perp\times\mathbf{B}}_{\text{ambipolar Poynting}} + \underbrace{\mathbf{F}_{\rm rad}}_{\text{radiative}}\Big] = \underbrace{-\rho\,\mathbf{v}\cdot\nabla\Phi}_{\text{gravity}}\end{aligned}} \tag{33}$$

This is Eq. (30) of the previous version written in full. Equivalent compact form (Eq. 1 of the 2020 paper): with $P^* = (P + \frac{B^2}{2})\mathsf{I}$ and the identity $(E+P^*)\mathbf{v} - \mathbf{B}(\mathbf{v}\cdot\mathbf{B}) = \big[\frac{\gamma}{\gamma-1}P + \frac{1}{2}\rho v^2\big]\mathbf{v} + \mathbf{B}\times(\mathbf{v}\times\mathbf{B})$, the ideal–inviscid part of the flux collapses to $\mathbf{A} = \big[\frac{\gamma}{\gamma-1}P + \frac{1}{2}\rho v^2\big]\mathbf{v} + \mathbf{B}\times(\mathbf{v}\times\mathbf{B})$, i.e. Eq. (7) of that paper. In the radiation-MHD formulation the term $\nabla\cdot\mathbf{F}_{\rm rad}$ is replaced by the mixed-frame source $+\,cS_r(E)$ on the right-hand side (with the momentum source $-\boldsymbol{S}_r(P)$ handling the $\mathbf{v}\cdot\boldsymbol{S}_r(P)$ exchange internally); in the grey diffusion limit the two forms coincide.

Where are the non-ideal terms in Eq. (33)? Ask the class this deliberately, because the answer is the whole point. The *heating* terms $\eta_{\rm O}J^2$ and $\eta_{\rm A}J_\perp^2$ do *not* appear — and must not: they enter the magnetic equation (28) with a minus sign and the internal equation (32) with a plus sign, and cancel identically in the sum. In the total energy budget, non-ideal MHD survives *only through the flux*: the three non-ideal Poynting terms $\eta_{\rm O}\mathbf{J}\times\mathbf{B}$, $-\eta_{\rm H}B\,\mathbf{J}_\perp$, and $\eta_{\rm A}\mathbf{J}_\perp\times\mathbf{B}$ inside the divergence. Their size relative to the ideal Poynting flux is

$$\frac{|\eta\,\mathbf{J}\times\mathbf{B}|}{|v_K B_R B_\phi|} \sim \frac{\eta\,B^2/H}{v_K B^2} = \frac{\eta}{v_K H} \sim \mathrm{Rm}^{-1}, \tag{34}$$

the inverse magnetic Reynolds number: negligible wherever the gas is well coupled ($\mathrm{Rm} \gg 1$), and competitive only in dead zones ($\Lambda_{\rm O} \lesssim 1$) — where, however, the turbulence and hence the ideal flux are quenched too. So to leading order in Rm^{-1}, non-ideal physics changes *where* energy thermalizes (§5.6) and how magnetic flux is transported, but leaves the averaged energy fluxes A_R, A_z below untouched.

Bookkeeping theorem (state it explicitly): $Q_{\rm visc}$, $\eta_{\rm O}J^2$ and $\eta_{\rm A}J_\perp^2$ never appear in Eq. (33) — they are *internal conversions*, not sources. The only true sink is radiation; the only true source is gravity. Dissipation physics decides *where* and *through which channel* orbital energy becomes heat, not *how much* energy there is. This is why the total-energy route (§5.5) and the internal-energy route (dissipation-function route) must give the same $T_{\rm eff}$ — a useful consistency check for students.

5.4 Azimuthally averaged energy fluxes: deriving A_R and A_z

Average Eq. (33) and write it as $\partial_t\langle E\rangle = -\frac{1}{R}\partial_R(R\langle A_R\rangle) - \partial_z\langle A_z\rangle - \langle Q_{\rm cool}\rangle + \langle\mathbf{F}\cdot\mathbf{v}\rangle$. Insert $v_\phi = v_K + \delta v_\phi$ and keep leading order:

$$\begin{aligned}
\tfrac{1}{2}\rho v^2 v_R \;&\simeq\; \tfrac{1}{2}\rho v_K^2\, v_R + \rho v_K\,\delta v_\phi\, v_R && \text{(Keplerian KE advection + Reynolds work)}\\
\big[B^2\mathbf{v} - (\mathbf{v}\cdot\mathbf{B})\mathbf{B}\big]_R \;&\simeq\; -\,v_K B_R B_\phi && \text{(Maxwell work / ideal Poynting)}\\
(\Pi\cdot\mathbf{v})_R = \Pi_{R\phi}\,v_\phi \;&\simeq\; v_K\,\Pi_{R\phi} && \text{(viscous work),}
\end{aligned} \tag{35}$$

with the non-ideal Poynting corrections suppressed by Rm^{-1} (the estimate in §5.3). The three "work" terms assemble into $v_K T_{R\phi}$ with the *same* total stress as in the angular momentum equation:

$$\boxed{A_R = \frac{\gamma}{\gamma-1}P\,v_R + \frac{1}{2}\rho v_R v_K^2 + v_K\,T_{R\phi}, \qquad A_z = \frac{\gamma}{\gamma-1}P\,v_z + \frac{1}{2}\rho v_z v_K^2 + v_K\,T_{\phi z}.} \tag{36}$$

This is Eqs. (13)–(14) of the 2020 paper, now with the viscous channel explicit. The structural statement: *angular momentum flux RT and energy flux $\Omega\cdot RT = v_K T$ are carried by the same stress* — torque times angular velocity, exactly as for gears.

5.5 The energy-extraction identity and the steady-state cooling rate

Split the divergence of the stress-work flux. With $v_K = R\Omega$ and the product rule, $\frac{1}{R}\partial_R(R^2\Omega T_{R\phi}) = \frac{\Omega}{R}\partial_R(R^2 T_{R\phi}) + T_{R\phi} R\,\partial_R\Omega$, i.e.

$$\frac{1}{R}\frac{\partial}{\partial R}\big(R\,v_K T_{R\phi}\big) = \underbrace{\frac{\Omega}{R}\frac{\partial\big(R^2\,T_{R\phi}\big)}{\partial R}}_{\substack{\Omega\times(\text{divergence of the AM flux } R\,T_{R\phi})\\ \text{torque work: travels with the flow}}} + \underbrace{T_{R\phi}\,R\frac{\partial\Omega}{\partial R}}_{\text{extracted from the shear}} . \tag{37}$$

The second term, $-\frac{3}{2}\Omega\,T_{R\phi}$ per unit volume for Keplerian rotation, is the rate at which *orbital* energy is converted. Channel by channel: **viscous** — equals $Q_{\rm visc}$ identically, extraction and thermalization coincide; **Reynolds** — feeds turbulent kinetic energy, thermalized within an eddy turnover, i.e. locally to within $\sim H$; **Maxwell** — feeds magnetic energy, which travels as Poynting flux (Eq. 30) before dissipating as $\eta_{\rm O}J^2$ in current sheets elsewhere, or never thermalizes and leaves with the wind (Eq. 31). The angular momentum budget is blind to these distinctions; the energy budget is not. *This is the single most important sentence of the lecture.*

Assume steady state, ignore the pressure term in A_R, set $v_z \simeq 0$ in A_z, and vertically integrate. Substituting the angular-momentum solution $W_{R\phi} = \dot{M}v_K/2\pi R - C/R^2$ into $\langle A_R\rangle$:

$$2\pi\langle Q_{\rm cool}\rangle^{(\rm int)} = \underbrace{-\frac{1}{2}\frac{\dot{M}v_K^2}{R^2}}_{\substack{\text{radial derivative of}\\ \text{Keplerian KE flux}}} + \underbrace{\frac{\dot{M}v_K^2}{R^2} - \frac{3}{2}\frac{C\,v_K}{R^3}}_{\substack{\text{radial derivative of}\\ \text{stress work } v_K T_{R\phi}}} + \underbrace{\frac{\dot{M}v_K^2}{R^2}}_{\substack{\text{gravitational}\\ \text{release}}} . \tag{38}$$

With the zero-torque boundary $C = \dot{M}R_{\rm in}v_{K,\rm in}/2\pi$:

$$\boxed{\sigma T_{\rm eff}^4 = \frac{3GM\dot{M}}{8\pi R^3}\left(1-\sqrt{\frac{R_{\rm in}}{R}}\right).} \tag{39}$$

Cross-check via the dissipation route: $D(R) = \int(Q_{\rm visc} + \eta_{\rm O}J^2 + \eta_{\rm A}J_\perp^2 + \varepsilon_{\rm turb})\,{\rm d}z = \frac{3}{2}\Omega\,W_{R\phi}$ gives the same result — as the bookkeeping theorem demands — *provided all extracted energy thermalizes locally in radius.* It does not matter whether the joules come from molecular viscosity, turbulent cascade, or Ohmic current sheets; that is why the classic $T_{\rm eff}$ profile survived from Shakura–Sunyaev to the MRI era.

The factor-of-3 bookkeeping. At $R \gg R_{\rm in}$ the local gravitational release is $\dot{M}v_K^2/2\pi R^2$ per unit area (both faces), yet the disc radiates $3\dot{M}v_K^2/4\pi R^2$ — three times the net local liberation, because half of the released potential energy is stored back as increased orbital kinetic energy. The excess is delivered by the stress-work flux $v_K T_{R\phi}$: the inner disc does work on the outer disc. Global check:

$$L = \int_{R_{\rm in}}^{\infty} 2\pi R\,D(R)\,{\rm d}R = \frac{3GM\dot{M}}{2}\int_{R_{\rm in}}^{\infty}\frac{1-\sqrt{R_{\rm in}/R}}{R^2}\,{\rm d}R = \frac{GM\dot{M}}{2R_{\rm in}}, \tag{40}$$

half the binding energy at $R_{\rm in}$; the other half arrives at the inner edge as kinetic energy (the boundary-layer / magnetospheric problem).

5.6 Vertical structure of the heating

Where the heat is deposited differs radically between channels, even at equal $T_{\rm eff}$:

- **Viscous (α) model:** $Q_+ \propto \rho\nu\Omega^2 \propto \rho$ — heating tracks density, i.e. is midplane-loaded. With the two-stream grey solution the interior temperature is

$$T^4(\tau) = \frac{3}{4}\,T_{\rm eff}^4\left[\tau\Big(1-\frac{\tau}{\tau_{\rm tot}}\Big)+\frac{1}{3}\right], \tag{41}$$

  so $T_{\rm mid}^4 \sim \frac{3}{16}\tau_{\rm tot}T_{\rm eff}^4$: a hot midplane.

- **MRI turbulence:** the stress (and hence extraction) is roughly uniform with height or even surface-loaded, because $T_{R\phi}$ does not track ρ (Figs. 7–9 of the 2018 paper); moreover turbulence itself advects heat. The simulated midplane sits a factor $\sim$3 cooler than the equal-flux α model.

- **Ohmic:** $\eta_{\rm O}J^2$ is concentrated in current sheets. But in poorly ionized protoplanetary discs there is a self-limiting twist: MRI requires the Elsasser number

$$\Lambda_{\rm O} \equiv \frac{v_{Az}^2}{\eta_{\rm O}\Omega} \gtrsim 1 \qquad \text{(derived, with its siblings Ha, Am, in Appendix A.7)}, \tag{42}$$

  so precisely where $\eta_{\rm O}$ is large (the midplane dead zone, Gammie 1996) the turbulence — and with it the currents — is quenched. Ohmic heating is therefore significant in the marginal layers ($\Lambda_{\rm O} \sim 1$), not deep in the dead zone; the dead-zone midplane is heated mainly by radiative diffusion from the active layers above. This is the energetic side of layered accretion.

Punchline for the class: two discs with identical $\dot{M}$ and identical $T_{\rm eff}(R)$ can have midplane temperatures differing by factors of several, depending on the dissipation channel — with direct consequences for chemistry, ice lines, ionization (hence back onto $\eta_{\rm O}$ itself: a genuine feedback loop), and the thermal-instability question for outbursts.

6 Interlude: what the wind needs from us

Sections 4.2 and 5 used four facts about magnetized winds without proof: the invariants k and ω; the angular momentum invariant $l = \omega R_A^2$; and the Bernoulli invariant e whose Poynting part is $-\omega R B_\phi/k$ per unit mass. We now derive all four from steady ideal MHD. This is the Weber & Davis (1967) theory, extended from the equatorial plane to poloidal field lines (Mestel 1968; Blandford & Payne 1982).

7 Steady axisymmetric MHD winds: deriving the Weber–Davis invariants

Assumptions: steady ($\partial_t = 0$), axisymmetric ($\partial_\phi = 0$), *ideal* MHD ($\mathbf{E} = -\mathbf{v}\times\mathbf{B}$), inviscid. Everything below is exact under these assumptions.

7.1 Step 0: the flux function — why $\mathbf{B}_p = \frac{1}{R}\nabla\psi\times\hat{\boldsymbol{\phi}}$ is fully general

(a) Counting. The poloidal field has two components, $B_R(R,z)$ and $B_z(R,z)$. In axisymmetry the solenoidal condition reads

$$\nabla\cdot\mathbf{B} = \frac{1}{R}\frac{\partial(RB_R)}{\partial R} + \underbrace{\frac{1}{R}\frac{\partial B_\phi}{\partial\phi}}_{=0} + \frac{\partial B_z}{\partial z} = 0, \tag{43}$$

one constraint on two functions — so the poloidal field contains only *one* free scalar. The flux function $\psi(R,z)$ is that scalar:

$$\boxed{\mathbf{B}_p = \frac{1}{R}\,\nabla\psi\times\hat{\boldsymbol{\phi}} \qquad\Longleftrightarrow\qquad B_R = -\frac{1}{R}\frac{\partial\psi}{\partial z},\quad B_z = +\frac{1}{R}\frac{\partial\psi}{\partial R},} \tag{44}$$

using $\hat{\mathbf{R}}\times\hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}$ and $\hat{\mathbf{z}}\times\hat{\boldsymbol{\phi}} = -\hat{\mathbf{R}}$. (Note it is the gradient of the scalar ψ, crossed with the azimuthal unit vector $\hat{\boldsymbol{\phi}}$ — not a gradient of the coordinate ϕ.)

(b) $\nabla\cdot\mathbf{B}_p = 0$ is automatic — and the converse holds. Substitute the components:

$$\nabla\cdot\mathbf{B}_p = \frac{1}{R}\frac{\partial}{\partial R}\Big(-\frac{\partial\psi}{\partial z}\Big) + \frac{\partial}{\partial z}\Big(\frac{1}{R}\frac{\partial\psi}{\partial R}\Big) = \frac{1}{R}\Big(-\frac{\partial^2\psi}{\partial R\,\partial z} + \frac{\partial^2\psi}{\partial z\,\partial R}\Big) = 0 \tag{45}$$

identically, by equality of mixed partials — for *any* ψ. Conversely, the axisymmetric condition $\partial_R(RB_R) = -\partial_z(RB_z)$ is exactly the integrability condition guaranteeing that a ψ exists with $RB_R = -\partial_z\psi$ and $RB_z = +\partial_R\psi$. The representation is therefore not an ansatz but fully general for axisymmetric solenoidal fields. (Equivalently, write $\mathbf{B}_p = \nabla\times(A_\phi\hat{\boldsymbol{\phi}})$ and identify $\psi = RA_\phi$; the vector-potential form and the $\nabla\psi\times\hat{\boldsymbol{\phi}}$ form are the same identity.)

(c) Geometry: field lines are level curves of ψ. $\nabla\psi$ points *across* the contours of ψ in the (R,z) plane; crossing with $\hat{\boldsymbol{\phi}}$ rotates it by 90° within that plane, so $\mathbf{B}_p$ is everywhere *tangent* to the curves $\psi = \text{const}$:

$$\mathbf{B}_p\cdot\nabla\psi = \frac{1}{R}\,(\nabla\psi\times\hat{\boldsymbol{\phi}})\cdot\nabla\psi = 0. \tag{46}$$

This is the 2D stream-function trick of incompressible hydrodynamics ($\mathbf{v} = \nabla\times(\Psi\hat{\mathbf{z}})$, streamlines $= \Psi$ contours) adapted to axisymmetry; the $1/R$ is the metric factor that adaptation requires.

(d) Physical meaning of the normalization. With the $1/R$ included, ψ is the poloidal magnetic flux through a circle of radius R at height z, divided by 2π (taking $\psi = 0$ on the axis):

$$\Phi_B(R,z) = \int_0^R B_z\,2\pi R'\,\mathrm{d}R' = 2\pi\int_0^R \frac{\partial\psi}{\partial R'}\,\mathrm{d}R' = 2\pi\,\psi(R,z) \tag{47}$$

— hence the name *flux function*: ψ counts the enclosed flux.

(e) Why it matters for everything below. Because field lines are $\psi = \text{const}$ curves, "constant along a field line" becomes the precise statement "a function of ψ alone" — this is what makes the invariants $k(\psi), \omega(\psi), l(\psi), e(\psi)$ well-defined objects. The inverse rotation of the definition,

$$\hat{\boldsymbol{\phi}}\times\mathbf{B}_p = \frac{\nabla\psi}{R}, \tag{48}$$

is used repeatedly below; in particular it is what lets Step 2 conclude $\mathbf{E}\propto\nabla\psi$ and hence $\Phi_E = \Phi_E(\psi)$ — the entire invariant structure of the Weber–Davis theory rests on this construction. One caveat: the construction says nothing about B_ϕ, which is unconstrained by $\nabla\cdot\mathbf{B} = 0$ in axisymmetry and needs its own function — in the wind problem it is delivered by the invariants instead (Eq. 59).

7.2 Step 1: $\mathbf{v}_p \parallel \mathbf{B}_p$ and the mass-loading invariant k

Steady Faraday: $\nabla\times\mathbf{E} = 0$, so $\mathbf{E} = -\nabla\Phi_E$ for some potential. Axisymmetry gives $E_\phi = -\frac{1}{R}\partial_\phi\Phi_E = 0$. But

$$E_\phi = -(\mathbf{v}\times\mathbf{B})\cdot\hat{\boldsymbol{\phi}} = -(\mathbf{v}_p\times\mathbf{B}_p)\cdot\hat{\boldsymbol{\phi}}, \tag{49}$$

so $\mathbf{v}_p \times \mathbf{B}_p = 0$: *the poloidal velocity is parallel to the poloidal field.* Write $\mathbf{v}_p = \dfrac{k}{\rho}\mathbf{B}_p$ for some scalar field k. Mass conservation then pins k down:

$$0 = \nabla \cdot (\rho \mathbf{v}_p) = \nabla \cdot (k\,\mathbf{B}_p) = \mathbf{B}_p \cdot \nabla k \quad \Longrightarrow \quad \boxed{k = \frac{\rho\, v_p}{B_p} = k(\psi)} \tag{50}$$

k is the mass flux per unit magnetic flux — the *mass loading* of the field line.

7.3 Step 2: isorotation and the invariant ω

Now the poloidal part of $\mathbf{E} = -\nabla\Phi_E$. Compute $\mathbf{E}$ with $\mathbf{v}_p = (k/\rho)\mathbf{B}_p$:

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B} = -\big(v_\phi \hat{\boldsymbol{\phi}} + \mathbf{v}_p\big) \times \big(B_\phi \hat{\boldsymbol{\phi}} + \mathbf{B}_p\big) = -\Big[v_\phi - \frac{k}{\rho}B_\phi\Big]\,\hat{\boldsymbol{\phi}} \times \mathbf{B}_p = -\Big[v_\phi - \frac{k}{\rho}B_\phi\Big]\,\frac{\nabla\psi}{R}. \tag{51}$$

For this to equal $-\nabla\Phi_E$ we need $\Phi_E = \Phi_E(\psi)$ and

$$\boxed{\omega(\psi) \equiv \frac{\mathrm{d}\Phi_E}{\mathrm{d}\psi} = \frac{v_\phi}{R} - \frac{kB_\phi}{\rho R} = \Omega - \frac{kB_\phi}{\rho R}} \tag{52}$$

constant along each field line: **Ferraro isorotation**. Physically ω is the angular velocity of the magnetic field line, "anchored" at its footpoint; in the disc, where ρ is large and the second term negligible, $\omega \simeq \Omega_\mathrm{K}(R_0)$ of the launch radius. (The simulations find $\omega \approx 1.5\,\Omega_K$ because the launch point sits at $z \sim 1.5R$ where the equipotential geometry shifts the effective footpoint — Fig. 16 of the 2018 paper.)

7.4 Step 3: the angular momentum invariant l

The ϕ-component of the momentum equation, using the identity $(\mathbf{J} \times \mathbf{B})_\phi = \frac{1}{R}\mathbf{B}_p \cdot \nabla(RB_\phi)$ and steady axisymmetry:

$$\rho\,\mathbf{v}_p \cdot \nabla(Rv_\phi) = \mathbf{B}_p \cdot \nabla(RB_\phi). \tag{53}$$

Substitute $\rho\mathbf{v}_p = k\mathbf{B}_p$ and pull k (constant along $\mathbf{B}_p$) inside:

$$\mathbf{B}_p \cdot \nabla\Big[k\,Rv_\phi - RB_\phi\Big] = 0 \quad \Longrightarrow \quad \boxed{l(\psi) = R\Big(v_\phi - \frac{B_\phi}{k}\Big)} \tag{54}$$

The specific angular momentum of the outflow is carried in *two* forms: matter (Rv_ϕ) and magnetic torque ($-RB_\phi/k$ per unit mass flux). Near the base almost all of l is magnetic; far away it has been converted to matter — the wind is a device for paying out angular momentum along field lines.

7.5 Step 4: the Bernoulli invariant e

Take the steady total energy equation (33) in the ideal, inviscid, adiabatic limit and integrate over a thin flux tube of cross-section $\propto 1/B_p$. The advective flux per unit mass flux gives $\frac{1}{2}v^2 + h + \Phi$ (with enthalpy $\mathrm{d}h = \mathrm{d}P/\rho$); the Poynting contribution is, from Eq. (31),

$$\frac{\mathbf{S}_p \cdot \mathbf{B}_p/B_p}{\rho v_p} = \frac{-\,\omega R\,B_\phi\,B_p}{\rho v_p} = -\,\frac{\omega RB_\phi}{k}. \tag{55}$$

Hence

$$\boxed{e(\psi) = \frac{1}{2}v^2 + h + \Phi - \frac{\omega R B_\phi}{k} \qquad \text{constant along field lines.}} \tag{56}$$

Equivalently $e = \frac{1}{2}v^2 + h + \Phi + \big(B_\phi B_\phi/\rho - B_\phi v_\phi/k\big)$ — Eqs. (21)–(23) of the 2018 paper. A "cold" wind is one with $h \ll$ the magnetic term at the base: the energy source is Poynting flux, i.e. shear-extracted Maxwell-stress energy that never thermalized (§5).

7.6 Step 5: the algebra, the Alfvén point, and $l = \omega R_A^2$

The invariants ω and l are two linear equations for the two unknowns (v_ϕ, B_ϕ) at each point of a field line:

$$v_\phi = \omega R + \frac{k}{\rho}B_\phi, \qquad v_\phi = \frac{l}{R} + \frac{B_\phi}{k}. \tag{57}$$

Subtract, and introduce the **poloidal Alfvén Mach number**

$$M_A^2 \equiv \frac{v_p^2}{v_{Ap}^2} = \frac{v_p^2\rho}{B_p^2} = \frac{k^2}{\rho} \qquad \big(\text{using } v_p = kB_p/\rho\big): \tag{58}$$

$$B_\phi\Big(\frac{k}{\rho} - \frac{1}{k}\Big) = \frac{l}{R} - \omega R \quad\Longrightarrow\quad B_\phi = \frac{k}{R}\,\frac{l - \omega R^2}{M_A^2 - 1}, \qquad v_\phi = \frac{\omega R\,\big(M_A^2\, l/\omega R^2 - 1\big)}{M_A^2 - 1}. \tag{59}$$

Both expressions have a vanishing denominator where $M_A = 1$ — the **Alfvén point** $R = R_A$, which every escaping wind must cross. A physical (finite) solution requires the numerator to vanish there too:

$$\boxed{l = \omega R_A^2.} \tag{60}$$

This is the celebrated result: *the wind removes angular momentum as if the gas corotated rigidly out to the Alfvén radius*, giving the lever arm $\lambda = (R_A/R_0)^2$ used in §4.2. (The full problem has three critical points — slow magnetosonic, Alfvén, fast magnetosonic; regularity at all three fixes the mass loading k as an eigenvalue. The 2018 simulation domain contains all three surfaces, Figs. 4–5, which is what makes the measured invariants meaningful.)

7.7 Step 6: asymptotics and observational diagnostics

Terminal speed. For a cold wind with $\lambda \gg 1$, the base energy is almost entirely Poynting: $e \approx \omega l = \omega^2 R_A^2$. Far downstream the Poynting term decays and $e \to \frac{1}{2}v_{p,\infty}^2$, so

$$v_{p,\infty} \simeq \sqrt{2}\,\omega R_A, \tag{61}$$

several times the footpoint Keplerian speed — matching the $\sim$300–500 km s^{-1} terminal speeds in the FU Ori simulations ($v_K \sim 100$ km s^{-1} at the inner boundary).

Launch-point diagnostic. The combination $J \equiv e - \omega l = \frac{1}{2}v^2 + h + \Phi - \omega R v_\phi$ (a Jacobi-type constant in the frame rotating at ω) is also invariant. At the base $J \approx \frac{1}{2}v_K^2 - v_K^2 - v_K^2 = -\frac{3}{2}v_K^2$; far away, with $v_{p,\infty} \gg v_{\phi,\infty}$ and $\Phi, h \to 0$, $J \approx \frac{1}{2}v_{p,\infty}^2 - \omega R_\infty v_{\phi,\infty}$. If $v_{p,\infty} \gg v_K$ the $-\frac{3}{2}v_K^2$ is negligible and

$$\boxed{v_{p,\infty}^2 \simeq 2\,\omega\, R_\infty v_{\phi,\infty} = 2\, R_\infty v_{\phi,\infty}\sqrt{\frac{GM}{R_0^3}} \quad\Longrightarrow\quad R_0 \simeq \Big(\frac{2\, R_\infty v_{\phi,\infty}}{v_{p,\infty}^2}\Big)^{2/3}(GM)^{1/3}.} \tag{62}$$

(Anderson et al. 2003.) Measuring the poloidal and rotation speeds of a resolved outflow at large distance locates the launch radius — the diagnostic applied to TMC1A by Bjerkeli et al. (2016), and testable against the simulations: plugging $v_p = 4$, $v_\phi = 1.2$, $R = 6$ at $r = 30$ from the 2018 run returns $R_0 \simeq 1.1$, against the true launch point $R_0 = 1$.

Wind torque, closing the loop. With $l = \omega R_A^2$ in hand, the surface flux follows in one line. The *full* angular-momentum flux through the wind base is $\rho v_z v_\phi - B_z B_\phi = \rho v_z\big(v_\phi - B_\phi/k\big) = \rho v_z\, l/R = \rho v_z\, \omega R_A^2/R$, using $kB_z = \rho v_z$ along the poloidal line. The *stress of Eq. (16)* — the excess over the Keplerian advection already booked into $\dot M(R)$ — is $T_{\phi z}\big|_{\rm base} = \rho v_z(v_\phi - v_K) - B_z B_\phi = \rho v_z\,(l-j)/R$, the form used in §4.2. Same bookkeeping split as Remark 3: the wind *exports* l per gram; the disc is *braked* by $l - j$ per gram. For $\lambda \sim 10$ the distinction is a 10% quibble; for the $\lambda \lesssim 2$ magneto-thermal winds argued to be generic by Bai (2026, §4.5.3), it is order unity — a $\lambda = 1.5$ wind delivers only a third of the naive $\rho v_z\, l/R$ torque to the disc.

8 The wind-modified energy budget

Once the corona extends to $z \sim R$, spherical-polar coordinates are the natural frame. The steady energy balance reads (Eq. 28 of Zhu, Jiang & Stone 2020, with T the full stress)

$$\langle Q_{\rm cool}\rangle = -\frac{1}{r^2}\frac{\partial}{\partial r}\Big(-\tfrac12\dot{\mathcal{M}}\,v_K^2 + r^2 v_K T_{r\phi}\Big) - \frac{1}{r\sin\theta}\frac{\partial\big(\sin\theta\, v_K T_{\theta\phi}\big)}{\partial\theta} + \langle \mathbf{F}\cdot\mathbf{v}\rangle. \tag{63}$$

The θ-flux term is what the thin-disc derivation discards. Per unit wind mass, the Bernoulli invariant of §7 gives an exported energy

$$e \approx \omega\, l = \lambda\,\Omega^2 R^2 \approx \lambda\, v_K^2\, : \tag{64}$$

for $\lambda \sim 10$, each gram of wind carries away many times its own binding energy, all of it drawn from the shear via the Poynting channel of §5.2 (Eq. 31) — Maxwell-stress energy that was extracted but *never thermalized*. Even a mass-loss rate of a few per cent of $\dot M$ is therefore energetically first-order. The FU Ori radiation-MHD simulations measure the split directly: the radial advection terms nearly cancel, and the wind flux removes about half the liberated gravitational energy, leaving

$$\boxed{\langle Q_{\rm cool}\rangle \simeq \frac{\dot{\mathcal{M}}\,v_K^2}{2r^2} \quad\Longrightarrow\quad \sigma T_{\rm eff}^4 = \frac{GM\dot M}{8\pi R^3},} \tag{65}$$

a factor $\sim$3 below Eq. (39) at equal $\dot M$. Observational corollary: SED-inferred accretion rates are underestimates by a factor $\sim$2–3 when a magnetized wind operates.

9 Summary board

Angular momentum	$T_{R\phi}$ = viscous + Reynolds + Maxwell enter *identically*; non-ideal terms do not appear at all; α is a closure; molecular $\alpha \sim 10^{-12}$ vs. required $\sim 10^{-2}$; Keplerian flow is Rayleigh-stable — the MRI supplies the correlation.
Wind invariants	$k = \rho v_p/B_p$ (mass loading, from $\mathbf{v}_p \parallel \mathbf{B}_p$ + continuity); ω (isorotation, from $E_\phi = 0$); $l = R(v_\phi - B_\phi/k)$ (ϕ-momentum); $e = \frac{1}{2}v^2 + h + \Phi - \omega R B_\phi/k$ (energy + Poynting). Alfvén regularity $\Rightarrow l = \omega R_A^2$; exported l per gram $= \lambda v_K R_0$, disc braking $= (\lambda - 1)v_K R_0$ (the gram's own j leaves for free); $v_{p,\infty} \simeq \sqrt{2}\,\omega R_A$.
Poynting flux	$\mathbf{S} = \mathbf{E}\times\mathbf{B} = B^2\mathbf{v} - (\mathbf{v}\cdot\mathbf{B})\mathbf{B} + \eta_{\rm O}\mathbf{J}\times\mathbf{B} - \eta_{\rm H}B\mathbf{J}_\perp + \eta_{\rm A}\mathbf{J}_\perp\times\mathbf{B}$; in the disc $S_R \simeq -v_K B_R B_\phi$ (stress $\times$ v_K); in the wind $\mathbf{S}_p = -\omega R B_\phi \mathbf{B}_p$. Hall transports without dissipating.
Energy channels	Kinetic loses $Q_{\rm visc}$ and exchanges $\mathbf{v}\cdot(\mathbf{J}\times\mathbf{B})$ with magnetic; magnetic loses $\eta_{\rm O}J^2 + \eta_{\rm A}J_\perp^2$ and transports via Poynting flux. All dissipation terms cancel in the total (Eq. 33) — gravity in, radiation out.
Where the heat lands	Viscous: dissipates where it extracts (midplane-loaded, hot midplane). Turbulent/Maxwell: transports before dissipating (surface-loaded, cool midplane). Ohmic: current sheets, self-limited by $\Lambda_{\rm O} \gtrsim 1$ (dead zones). Wind: exports $\sim \lambda v_K^2$ per gram without thermalizing, halving the radiated luminosity.
Non-ideal microphysics (App. A)	Species force balance + one preferred direction $\hat{\mathbf{b}} \Rightarrow$ exactly three terms. Hall parameter β_s = gyro/collision frequency; $\lvert\beta_e\rvert \sim 10^3\beta_i$. Ladder $\eta_{\rm O} : \eta_{\rm H} : \eta_{\rm A} = 1 : \lvert\beta_e\rvert : \lvert\beta_e\rvert\beta_i$, i.e. $\propto (B/\rho)^{0,1,2}$; all $\propto x_e^{-1}$. $\mathbf{E}'\cdot\mathbf{J}$ = total interspecies friction $\Rightarrow$ Ohmic/ambipolar heat, Hall cannot. Midplane: Ohmic $\to$ Hall $\to$ ambipolar going outward; Am maps to x_e independent of B.
Punchline	The degeneracy between accretion models lives in the angular momentum equation and breaks in the energy equation — both in $T_{\rm eff}(R)$ and in the vertical heating profile.

A The generalized Ohm's law derived: why $\eta_{\rm O}\mathbf{J}$, $\eta_{\rm H}\,\mathbf{J}\times\hat{\mathbf{b}}$, $\eta_{\rm A}\mathbf{J}_\perp$

Equation (3) postulated three non-ideal terms with three coefficients. This appendix derives them from multifluid force balance, following Wardle & Ng (1999), Bai (2011a), and the pedagogical route of Bai (2026, §3.1.3). The payoff is threefold: (i) the *three-term structure* is forced by symmetry — with one preferred direction $\hat{\mathbf{b}}$, a linear relation between $\mathbf{E}'$ and $\mathbf{J}$ has exactly three independent pieces; (ii) each coefficient corresponds to a distinct pair of species being frozen together, controlled by a single dimensionless number per species (the Hall parameter); (iii) the energy statements used in §5.2 — Ohmic and ambipolar heat, Hall does not — become a one-line theorem about friction.

Units in this appendix. The microphysics involves individual charges gyrating and colliding, so we work in Gaussian units here (explicit e, c, 4π): a charge gyrates at $\omega_{cs} = \lvert Z_s\rvert eB/(m_s c)$, and

Ohm's law reads $\mathbf{E}' = \frac{4\pi}{c^2}\left[\eta_{\rm O}\mathbf{J}_{\rm G} + \eta_{\rm H}\mathbf{J}_{\rm G}\times\hat{\mathbf{b}} + \eta_{\rm A}\mathbf{J}_{{\rm G}\perp}\right]$ with $\mathbf{J}_{\rm G} = \frac{c}{4\pi}\nabla\times\mathbf{B}_{\rm G}$. The diffusivities η that come out are *physical diffusivities in* $cm^2\ s^{-1}$ — they enter the induction equation as $\partial_t\mathbf{B} \supset \eta\nabla^2\mathbf{B}$ — and are therefore unit-convention independent: the same numbers plug into the code-units Eq. (3) unchanged. Consistency check for the heating, since the notes claim it is $\eta_{\rm O}J^2$ with $\mathbf{J} = \nabla\times\mathbf{B}$: in Gaussian units the dissipation is $\mathbf{E}'\cdot\mathbf{J}_{\rm G} = \frac{4\pi}{c^2}\eta_{\rm O}J_{\rm G}^2 = \frac{4\pi}{c^2}\eta_{\rm O}\left(\frac{c}{4\pi}\right)^2|\nabla\times\mathbf{B}_{\rm G}|^2 = \eta_{\rm O}\,|\nabla\times(\mathbf{B}_{\rm G}/\sqrt{4\pi})|^2 = \eta_{\rm O}J^2$, using $\mathbf{B} = \mathbf{B}_{\rm G}/\sqrt{4\pi}$. The conventions mesh.

A.1 Step 0: why force balance (terminal drift) is exact enough

A protoplanetary disc is *weakly ionized*: ionization fractions $x_e \equiv n_e/n_n$ range from $\sim 10^{-16}$ at the dead-zone midplane to $\sim 10^{-10}$ in the bulk and $\gtrsim 10^{-5}$ only in the FUV-irradiated surface. Two consequences. First, the neutrals carry essentially all the mass and momentum: the "fluid" of the MHD equations *is* the neutral fluid, and the charged species are trace populations riding on it. Second, each charged species reaches terminal velocity almost instantly. Its momentum equation in the frame comoving with the neutrals is

$$m_s n_s \frac{\mathrm{d}\mathbf{v}'_s}{\mathrm{d}t} = n_s Z_s e\Big(\mathbf{E}' + \frac{\mathbf{v}'_s}{c}\times\mathbf{B}\Big) - \gamma_s\rho\, m_s n_s\,\mathbf{v}'_s \;-\; \nabla P_s, \qquad \gamma_s \equiv \frac{\langle\sigma v\rangle_s}{m_s + m_n}, \tag{66}$$

where $\mathbf{v}'_s$ is the drift relative to the neutrals, $\mathbf{E}' = \mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}$ is the electric field in the neutral frame, $\langle\sigma v\rangle_s$ is the momentum-transfer rate coefficient with neutrals (electrons: $\langle\sigma v\rangle_e \approx 8.3\times 10^{-9}\max[1,(T/100\,\mathrm{K})^{1/2}]\ \mathrm{cm^3\,s^{-1}}$; ions: $\langle\sigma v\rangle_i \approx 1.9\times10^{-9}\ \mathrm{cm^3\,s^{-1}}$, essentially the temperature-independent Langevin rate; Draine et al. 1983), and $\gamma_s\rho$ is the collision frequency of one particle of species s with the neutral sea.

Drop the inertia: the drag stopping time is $t_s = 1/(\gamma_s\rho)$. At the 1-au midplane of the base model ($n_n \sim 10^{14}\,\mathrm{cm^{-3}}$), $\gamma_e\rho \approx \langle\sigma v\rangle_e\, n_n \approx 8\times10^5\,\mathrm{s^{-1}}$, i.e. $t_e \sim 10^{-6}$ s against a dynamical time $\Omega^{-1} \sim 5\times10^6$ s: thirteen orders of magnitude. Ions are slower by $\sim m_i/m_n$ but still absurdly fast. *Drop the pressure:* $\nabla P_s/\nabla P \sim x_s \lesssim 10^{-10}$, and the neutral pressure already lives in the bulk momentum equation. What remains is exact force balance — Eq. (18) of Bai (2026):

$$\boxed{Z_s e\Big(\mathbf{E}' + \frac{\mathbf{v}'_s}{c}\times\mathbf{B}\Big) = \gamma_s\rho\, m_s\,\mathbf{v}'_s.} \tag{67}$$

Everything below is linear algebra applied to this one equation, plus charge neutrality $\sum_s n_s Z_s = 0$ and the definition $\mathbf{J}_{\rm G} = e\sum_s n_s Z_s \mathbf{v}'_s$. (The mean neutral flow drops out of $\mathbf{J}$ by neutrality — only the *drifts* carry current, which is why the whole calculation can be done in the neutral frame.)

A.2 Step 1: the Hall parameter, and the drift of a single species

Balance in Eq. (67) is between the Lorentz force and neutral drag. Their ratio defines the (signed) **Hall parameter**

$$\boxed{\beta_s \equiv \frac{Z_s e B}{m_s c}\,\frac{1}{\gamma_s\rho} = \pm\frac{\text{gyrofrequency}}{\text{collision frequency}}.} \tag{68}$$

$|\beta_s| \gg 1$: frozen to $\mathbf{B}$ — the particle completes many gyrations between collisions and is tied to the field. $|\beta_s| \ll 1$: frozen to the neutrals — it is knocked off its orbit every fraction of a gyration and is dragged along with the gas. Because γ_e and γ_i are comparable while $m_e \ll m_i$,

$$\frac{|\beta_e|}{\beta_i} = \frac{m_i\gamma_i}{m_e\gamma_e} \approx \frac{\langle\sigma v\rangle_i}{(m_e/m_n)\langle\sigma v\rangle_e} \approx \frac{1.9\times10^{-9}}{\frac{1}{2.34\times1836}\,8.3\times10^{-9}} \approx 10^3: \tag{69}$$

the electrons magnetize a thousand times more easily than the ions. This single mass-hierarchy fact is the origin of the three regimes: sweeping B/ρ upward, first the electrons cross $|\beta_e| = 1$, then much later the ions cross $\beta_i = 1$, and each crossing hands the physics to a different term of Ohm's law.

Solving the force balance. Split $\mathbf{E}' = \mathbf{E}'_\parallel + \mathbf{E}'_\perp$ relative to $\hat{\mathbf{b}}$ and define the signed mobility $\mu_s \equiv Z_s e/(m_s \gamma_s \rho)$. The parallel part is trivial (no Lorentz force): $\mathbf{v}'_{s\parallel} = \mu_s \mathbf{E}'_\parallel$. For the perpendicular part, Eq. (67) reads

$$\mathbf{v}'_{s\perp} - \beta_s\, \mathbf{v}'_{s\perp} \times \hat{\mathbf{b}} = \mu_s \mathbf{E}'_\perp. \tag{70}$$

Here is a trick worth teaching: on the plane perpendicular to $\hat{\mathbf{b}}$, the operator $\mathsf{K} : \mathbf{u} \mapsto \mathbf{u} \times \hat{\mathbf{b}}$ satisfies $\mathsf{K}^2 = -1$ (rotate by 90° twice = reverse), so K behaves exactly like the imaginary unit and $(1 - \beta_s \mathsf{K})^{-1} = (1 + \beta_s \mathsf{K})/(1 + \beta_s^2)$. Hence

$$\boxed{\mathbf{v}'_{s\perp} = \frac{\mu_s}{1 + \beta_s^2}\Big(\mathbf{E}'_\perp + \beta_s\, \mathbf{E}'_\perp \times \hat{\mathbf{b}}\Big).} \tag{71}$$

Check the limits, because they *are* the physics:

- $|\beta_s| \ll 1$: $\mathbf{v}'_{s\perp} \to \mu_s \mathbf{E}'_\perp$ — drag-limited drift *along* $\mathbf{E}'$, magnetic field irrelevant. Different signs of Z_s drift oppositely $\Rightarrow$ net current: this is ordinary conduction.
- $|\beta_s| \gg 1$: $\mathbf{v}'_{s\perp} \to (\mu_s/\beta_s)\, \mathbf{E}'_\perp \times \hat{\mathbf{b}} = \dfrac{c}{B}\, \mathbf{E}' \times \hat{\mathbf{b}}$ — the $\mathbf{E} \times \mathbf{B}$ **drift**, *independent of charge, mass, and collision rate.* A fully magnetized species therefore carries no perpendicular current by itself: everything drifts together. Perpendicular current requires *differential* magnetization between species. Hold that thought; it is the Hall effect.

A.3 Step 2: the conductivity tensor — three terms by symmetry

Sum $\mathbf{J}_\mathrm{G} = e \sum_s n_s Z_s \mathbf{v}'_s$ over species using (71). Since the only available structure is $\hat{\mathbf{b}}$, the result must take the form

$$\boxed{\mathbf{J}_\mathrm{G} = \sigma_\mathrm{O}\, \mathbf{E}'_\parallel + \sigma_\mathrm{H}\, \hat{\mathbf{b}} \times \mathbf{E}'_\perp + \sigma_\mathrm{P}\, \mathbf{E}'_\perp,} \tag{72}$$

with the **Ohmic, Hall and Pedersen conductivities**

$$\sigma_\mathrm{O} = \frac{ec}{B} \sum_s n_s Z_s \beta_s, \qquad \sigma_\mathrm{H} = \frac{ec}{B} \sum_s \frac{n_s Z_s}{1 + \beta_s^2}, \qquad \sigma_\mathrm{P} = \frac{ec}{B} \sum_s \frac{n_s Z_s \beta_s}{1 + \beta_s^2}, \tag{73}$$

where we used $e n_s Z_s \mu_s = (ec/B)\, n_s Z_s \beta_s$ (each term ≥ 0 since $Z_s \beta_s = |Z_s \beta_s|$), and, for σ_H, converted the $\mathbf{E}' \times \hat{\mathbf{b}}$ coefficient $\frac{ec}{B} \sum n_s Z_s \beta_s^2/(1 + \beta_s^2)$ into the form above via charge neutrality: $\sum_s n_s Z_s \big[1 - \frac{\beta_s^2}{1+\beta_s^2}\big] = \sum_s \frac{n_s Z_s}{1+\beta_s^2}$ and $\hat{\mathbf{b}} \times \mathbf{E}' = -\mathbf{E}' \times \hat{\mathbf{b}}$. Note the different characters: σ_O and σ_P are sums of positive terms; σ_H is a sum of *signed* terms that survives only because different species have different β_s.

Ask the class: why must σ_H vanish if all species had the same β? — Because then $\sigma_\mathrm{H} = \frac{ec}{B(1+\beta^2)} \sum n_s Z_s = 0$ by neutrality. The Hall conductivity measures the *spread* in magnetization across species; in a disc it is carried almost entirely by the electron–ion pair in the window $\beta_i \ll 1 \ll |\beta_e|$. (Foreshadowing: when charged grains dominate the charge budget, the signed sum can flip sign — §A.8.)

A.4 Step 3: inversion — the diffusivities

Ohm's law wants $\mathbf{E}'(\mathbf{J})$, so invert (72). Parallel: $\mathbf{E}'_{\parallel} = \mathbf{J}_{\mathrm{G}\parallel}/\sigma_{\mathrm{O}}$. Perpendicular: with the K-as-i trick again, $\mathbf{J}_{\mathrm{G}\perp} = (\sigma_{\mathrm{P}} - \sigma_{\mathrm{H}}\mathsf{K})\mathbf{E}'_{\perp}$, so

$$\mathbf{E}'_{\perp} = \frac{\sigma_{\mathrm{P}}\,\mathbf{J}_{\mathrm{G}\perp} + \sigma_{\mathrm{H}}\,\mathbf{J}_{\mathrm{G}} \times \hat{\mathbf{b}}}{\sigma_{\perp}^2}, \qquad \sigma_{\perp}^2 \equiv \sigma_{\mathrm{P}}^2 + \sigma_{\mathrm{H}}^2. \tag{74}$$

Assemble, splitting $\mathbf{J}_{\mathrm{G}} = \mathbf{J}_{\mathrm{G}\parallel} + \mathbf{J}_{\mathrm{G}\perp}$ so that one coefficient multiplies the *full* current:

$$\boxed{\eta_{\mathrm{O}} = \frac{c^2}{4\pi\sigma_{\mathrm{O}}}, \qquad \eta_{\mathrm{H}} = \frac{c^2}{4\pi}\,\frac{\sigma_{\mathrm{H}}}{\sigma_{\perp}^2}, \qquad \eta_{\mathrm{A}} = \frac{c^2}{4\pi}\Big(\frac{\sigma_{\mathrm{P}}}{\sigma_{\perp}^2} - \frac{1}{\sigma_{\mathrm{O}}}\Big).} \tag{75}$$

Two structural lessons before any limit is taken. (i) The decomposition $\eta_{\mathrm{O}}\mathbf{J} + \eta_{\mathrm{H}}\mathbf{J}\times\hat{\mathbf{b}} + \eta_{\mathrm{A}}\mathbf{J}_{\perp}$ of Eq. (3) is now *derived*: with one preferred direction there are exactly three independent linear maps, and these are they. (ii) Ambipolar diffusion is precisely the *anisotropy of the resistivity*: $\eta_{\mathrm{A}} = \eta_{\perp} - \eta_{\parallel}$ with $\eta_{\perp} \equiv (c^2/4\pi)\sigma_{\mathrm{P}}/\sigma_{\perp}^2$ and $\eta_{\parallel} = \eta_{\mathrm{O}}$. When no species is magnetized ($\beta_s \to 0$: $\sigma_{\mathrm{H}} \to 0$, $\sigma_{\mathrm{P}} \to \sigma_{\mathrm{O}}$) the conductivity is isotropic and both η_{H} and η_{A} vanish identically — non-ideal MHD collapses to pure Ohmic resistivity, as it must.

A.5 Step 4: the electron–ion hierarchy and the three regimes

Now specialize to electrons ($Z = -1$, $\beta_e < 0$) and one ion species ($Z = +1$, $\beta_i > 0$), $n_e = n_i$, with $|\beta_e| \approx 10^3\beta_i$. Evaluating (73)–(75) and keeping the dominant terms (Bai 2026, Eq. 21):

$$\boxed{\eta_{\mathrm{O}} = \frac{cB}{4\pi e n_e}\,\frac{1}{\beta_i + |\beta_e|} \approx \frac{c^2 m_e\gamma_e\rho}{4\pi e^2 n_e}, \qquad \eta_{\mathrm{H}} \approx \frac{cB}{4\pi e n_e}, \qquad \eta_{\mathrm{A}} = \eta_{\mathrm{O}}|\beta_e|\beta_i \approx \frac{B^2}{4\pi\gamma_i\rho_i\rho}.} \tag{76}$$

Rather than grinding the algebra, derive each limit *physically* in one line each — this is the part to do on the board.

(i) Ohmic regime ($\beta_i \ll |\beta_e| \ll 1$; dense gas, weak field). Both species are glued to the neutrals; $\mathbf{B}$ is irrelevant to conduction, so the response is isotropic, $\mathbf{E}' = \mathbf{J}_{\mathrm{G}}/\sigma$, with the current carried by the more mobile electrons: $\sigma = n_e e^2/(m_e\gamma_e\rho)$, i.e. $\eta_{\mathrm{O}} = c^2 m_e\gamma_e\rho/(4\pi e^2 n_e)$. With $\gamma_e\rho \approx \langle\sigma v\rangle_e n_n$ this is a pure number over x_e:

$$\eta_{\mathrm{O}} = \frac{c^2 m_e\langle\sigma v\rangle_e}{4\pi e^2}\,\frac{1}{x_e} \approx \frac{2.3\times10^3}{x_e}\,\max\Big[1, \Big(\tfrac{T}{100\,\mathrm{K}}\Big)^{1/2}\Big]\ \mathrm{cm^2\,s^{-1}}, \tag{77}$$

plugging in c, m_e, e and $\langle\sigma v\rangle_e = 8.3\times10^{-9}\,\mathrm{cm^3\,s^{-1}}$. Independent of B and of ρ at fixed x_e — a genuinely *collisional* coefficient. It damps the total current isotropically: true diffusion, hence the $\eta_{\mathrm{O}}J^2$ heating of §5.2, concentrated in current sheets.

(ii) Hall regime ($\beta_i \ll 1 \ll |\beta_e|$; the wide middle). Electrons ride the field, ions ride the neutrals, so the current *is* the electron–ion drift: $\mathbf{v}'_e \approx -\mathbf{J}_{\mathrm{G}}/(en_e)$. The field, frozen to the electrons, obeys

$$\mathbf{E}' = -\frac{\mathbf{v}'_e}{c}\times\mathbf{B} = \frac{\mathbf{J}_{\mathrm{G}}\times\mathbf{B}}{c\,en_e} \qquad \Longrightarrow \qquad \eta_{\mathrm{H}} = \frac{cB}{4\pi en_e}. \tag{78}$$

Note what is *absent*: any collision rate. The Hall term is not a diffusivity at all but a frozen-in condition in disguise — frozen to the electron fluid instead of the bulk — which is why it is dispersive (whistler waves), does no work (§A.6), and is odd under $\mathbf{B} \to -\mathbf{B}$ (the term $\mathbf{J} \times \hat{\mathbf{b}}$ flips with $\hat{\mathbf{b}}$ while the other two are even): the origin of the aligned/anti-aligned dichotomy and the Hall-shear instability (Bai 2026, §§4.1.1, 5.1.1). A useful equivalent: the Hall length $l_{\mathrm{H}} \equiv \eta_{\mathrm{H}}/v_A = (c/\omega_{pi})\sqrt{\rho/\rho_i}$ — the ion inertial length, amplified by the neutral mass loading (Kunz & Lesur 2013); Hall physics dominates scales below l_{H}.

(iii) Ambipolar regime ($1 \ll \beta_i \ll |\beta_e|$; tenuous gas, strong field). Now electrons *and* ions are frozen to $\mathbf{B}$; the entire charged plasma drifts as one body through the neutral sea. The drift is set by balancing the Lorentz force on the plasma against ion–neutral friction (electron friction is down by $m_e\gamma_e/m_i\gamma_i \sim 10^{-3}$):

$$\frac{\mathbf{J}_{\mathrm{G}} \times \mathbf{B}}{c} = \gamma_i\rho_i\rho\, \Delta\mathbf{v}_{\mathrm{in}} \qquad \Longrightarrow \qquad \mathbf{E}' = -\frac{\Delta\mathbf{v}_{\mathrm{in}}}{c} \times \mathbf{B} = \frac{(\mathbf{J}_{\mathrm{G}} \times \mathbf{B}) \times (-\mathbf{B})}{c^2\gamma_i\rho_i\rho} = \frac{B^2\,\mathbf{J}_{\mathrm{G}\perp}}{c^2\gamma_i\rho_i\rho}, \tag{79}$$

using $(\mathbf{J} \times \mathbf{B}) \times \mathbf{B} = -B^2\mathbf{J}_\perp$. Hence $\eta_{\mathrm{A}} = B^2/(4\pi\gamma_i\rho_i\rho)$, and *only* $\mathbf{J}_\perp$ appears for a transparent reason: a parallel current exerts no Lorentz force, drives no drift, and feels no friction. The heating is literally the friction of that drift, $Q_{\mathrm{AD}} = \gamma_i\rho_i\rho|\Delta\mathbf{v}_{\mathrm{in}}|^2 = \eta_{\mathrm{A}}J_\perp^2$ — the microscopic content of the ambipolar remark in §5.2. (In the notes' units, drop the 4π's exactly as in the ambipolar bullet there.)

The ladder. Divide the boxed results:

$$\boxed{\eta_{\mathrm{O}} : \eta_{\mathrm{H}} : \eta_{\mathrm{A}} = 1 : |\beta_e| : |\beta_e|\beta_i, \qquad \eta_{\mathrm{O}} \propto \left(\frac{B}{\rho}\right)^0, \quad \eta_{\mathrm{H}} \propto \left(\frac{B}{\rho}\right)^1, \quad \eta_{\mathrm{A}} \propto \left(\frac{B}{\rho}\right)^2, \quad \text{all } \propto x_e^{-1}.} \tag{80}$$

(Check: $\eta_{\mathrm{H}}/\eta_{\mathrm{O}} = eB/(m_e c\gamma_e\rho) = |\beta_e|$; $\eta_{\mathrm{A}}/\eta_{\mathrm{H}} = en_eB/(c\gamma_i\rho_i\rho) = \beta_i$ using $\rho_i = n_e m_i$.) Each factor of B/ρ promotes one more species from neutral-tied to field-tied, and the regime boundaries sit exactly at $|\beta_e| = 1$ and $\beta_i = 1$. This is why the midplane transitions Ohmic → Hall → ambipolar going *outward* in radius (density drops), and Ohmic → ambipolar going *up* at fixed radius, exactly as in Fig. 2 of Bai (2026).

Worked example (base disc of Bai 2026, midplane $\beta_{\mathrm{plasma}} = 500$). A one-line estimator: $|\beta_e| = \omega_{ce}/(\langle\sigma v\rangle_e n_n) \approx 2.1 \times 10^{15}\, B(\mathrm{G})/n_n(\mathrm{cm}^{-3})$ (for $T \lesssim 100$ K), and $\beta_i \approx 10^{-3}|\beta_e|$. At 0.2 au: $\rho_{\mathrm{mid}} \approx 1.7 \times 10^{-8}\,\mathrm{g\,cm^{-3}}$ ($n_n \approx 4 \times 10^{15}$), $c_s \approx 1.4 \times 10^5\,\mathrm{cm\,s^{-1}}$, so $B = \sqrt{8\pi\rho c_s^2/500} \approx 4$ G and $|\beta_e| \approx 2$, $\beta_i \approx 2 \times 10^{-3}$: Ohmic-dominated (marginally), as in Bai's figure. At 3 au: $n_n \approx 10^{13}$, $B \approx 0.09$ G, $|\beta_e| \approx 20$, $\beta_i \approx 0.02$: squarely Hall-dominated. Push to 50 au and β_i crosses unity: ambipolar. *Three regimes in one disc, one decade of radius apart* — this is why protoplanetary discs are the hardest accretion discs.

A.6 Step 5: the energy theorem — dissipation is interspecies friction

Dot the force balance (67) with $\mathbf{v}_s'$: the magnetic term drops ($\mathbf{v}'\cdot(\mathbf{v}'\times\mathbf{B}) = 0$), leaving $n_sZ_se\,\mathbf{E}'\cdot\mathbf{v}_s' = \gamma_s\rho\,\rho_s|\mathbf{v}_s'|^2$ per unit volume. Sum over species:

$$\boxed{\mathbf{E}' \cdot \mathbf{J}_{\mathrm{G}} = \sum_s \gamma_s\rho\,\rho_s\,|\mathbf{v}_s'|^2 \;\geq\; 0:} \tag{81}$$

all non-ideal dissipation is frictional heating of species drifting through the neutrals — manifestly positive definite, no calculation of σ's required. Evaluating the same quantity from the tensor, $\mathbf{E}'\cdot\mathbf{J}_{\mathrm{G}} = J_{\mathrm{G}\|}^2/\sigma_{\mathrm{O}} + (\sigma_{\mathrm{P}}/\sigma_\perp^2)J_{\mathrm{G}\perp}^2 = \frac{4\pi}{c^2}\left[\eta_{\mathrm{O}}J_{\mathrm{G}}^2 + \eta_{\mathrm{A}}J_{\mathrm{G}\perp}^2\right]$ (using $\eta_\perp = \eta_{\mathrm{O}} + \eta_{\mathrm{A}}$ from Step 3) — precisely the two heating terms of Eq. (28), now understood microscopically: Ohmic heat is electron–neutral friction, ambipolar heat is ion–neutral friction. And the Hall term? Its contribution $\mathbf{J}\cdot(\mathbf{J}\times\hat{\mathbf{b}})=0$ vanishes *algebraically*, but (81) says why *physically*: in the Hall regime the electron motion is a pure $\mathbf{E}\times\mathbf{B}$ drift (Step 1, second limit), and an $\mathbf{E}\times\mathbf{B}$ drift is perpendicular to $\mathbf{E}'$ — it pays no frictional toll in the direction of the force. Frozen-in conditions do not dissipate; they transport. That is the microphysics behind the "Hall paradox" remark of §5.2: nonzero Hall Poynting flux, zero Hall heating.

A.7 Step 6: Elsasser numbers — when does non-ideal MHD matter?

A diffusivity matters when it can kill the fastest-growing MRI mode. The most unstable wavelength is $\lambda_{\mathrm{MRI}}\sim 2\pi v_A/\Omega$ (tension must compete with shear), the growth rate is $\sim\Omega$, and diffusion damps at rate $\eta k^2\sim\eta\,\Omega^2/v_A^2$. Their ratio is the **Elsasser number** of each effect:

$$\boxed{\frac{\text{growth}}{\text{damping}}=\frac{v_A^2}{\eta\,\Omega}\;\equiv\;\Lambda\ (\text{Ohmic}),\quad \mathrm{Ha}\ (\text{Hall}),\quad \mathrm{Am}\ (\text{ambipolar});\quad \text{they bite when }\lesssim 1.} \tag{82}$$

The $\Lambda_{\mathrm{O}}\gtrsim 1$ dead-zone criterion of §5.6 is the Ohmic member of this family. The ladder (80) fixes their scalings:

$$\Lambda\propto\frac{B^2x_e}{\rho},\qquad \mathrm{Ha}=\frac{eBn_e}{c\rho\,\Omega}\propto B\,x_e,\qquad \mathrm{Am}=\frac{\gamma_i\rho_i}{\Omega}\propto x_e\,\rho\quad(\text{independent of }B!). \tag{83}$$

The last is worth a beat: because $\eta_{\mathrm{A}}\propto B^2$ exactly cancels $v_A^2\propto B^2$, Am is a pure ionization diagnostic — "ion–neutral collisions per orbit" — which is why the outer-disc coupling is quoted as $\mathrm{Am}\sim 1$ without reference to field strength, and why with ambipolar diffusion the MRI prefers *weaker* fields (the field strength drops out of the coupling but not out of the tension). Physically: if a neutral parcel collides with ions only once per orbit, the field — however strong — cannot grip the gas on the dynamical time.

Closing the loop to the main text. To leading order the non-ideal terms never touched the angular momentum equation (§3: the Lorentz force is $\mathbf{J}\times\mathbf{B}$ whatever Ohm's law says), cancelled out of the total energy equation except through $\mathcal{O}(\mathrm{Rm}^{-1})$ flux corrections (§5.3), and yet decide *everything* environmental: where the MRI lives ($\Lambda,\mathrm{Ha},\mathrm{Am}\gtrsim 1$), where the heat lands (§5.6), whether the wind couples (the $\mathrm{Am}\sim 1$ surface layers), and which field polarity a disc prefers (Hall). Microphysics does not change the budget; it redistributes it.

A.8 Postscript: grains, and when the formulas bend

Everything above assumed electrons and one ion. Real discs add charged grains, which do two things. First, grains dominate recombination wherever $x_e\lesssim 3\times10^{-10}\,\frac{f_{\mathrm{d2g}}}{10^{-2}}\,\frac{\mu\mathrm{m}}{a}\,\frac{T}{100\,\mathrm{K}}$ (Bai 2026, Eq. 16), depressing x_e and raising all three $\eta\propto x_e^{-1}$ together. Second, when charged grains carry a significant share of the charge, the signed sum in σ_{H} (Step 2) gains slow, heavy, mostly negative carriers ($\beta_{\mathrm{gr}}\ll\beta_i$): the clean $\eta_{\mathrm{H}}\propto B$ and $\eta_{\mathrm{A}}\propto B^2$ scalings fail, and η_{H} can even change sign above a threshold field strength — the loophole in the "Hall polarity" story (Bai 2011b; Xu & Bai 2016). The three-term *structure* of Eq. (3), however, is symmetry, not chemistry: it survives any number of species. Only the coefficients renegotiate.

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